

Abelian-Higgs and Vortices from ABJM: towards a string realization of AdS/CMT

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ABSTRACT: We present ansätze that reduce the mass-deformed ABJM model to gauged Abelian scalar theories, using the fuzzy sphere matrices G^α . One such reduction gives a Toda system, for which we find a new type of nonabelian vortex. Another gives the standard Abelian-Higgs model, thereby allowing us to embed all the usual (multi-)vortex solutions of the latter into the ABJM model. By turning off the mass deformation at the level of the reduced model, we can also continuously deform to the massive ϕ^4 theory in the massless ABJM case. In this way we can embed the Landau-Ginzburg model into the AdS/CFT correspondence as a consistent truncation of ABJM. In this context, the mass deformation parameter μ and a field VEV $\langle\phi\rangle$ act as g and g_c respectively, leading to a well-motivated AdS/CMT construction from string theory. To further this particular point, we propose a simple model for the condensed matter field theory that leads to an approximate description for the ABJM abelianization. Finally, we also find some BPS solutions to the mass-deformed ABJM model with a spacetime interpretation as an M2-brane ending on a spherical M5-brane.

KEYWORDS: [ABJM model](#), [abelianization](#), [vortices](#), [M-branes](#).

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1. Introduction

Since its beginnings in 1997, the AdS/CFT correspondence [1] has found application in a variety of phenomena; not only in quantum gravity but also, increasingly in fields as diverse as low energy QCD and condensed matter. Its original formulation described four dimensional $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory with an $SU(N)$ gauge group in the large N limit from the perspective of a dual gravitational theory on $AdS_5 \times S^5$. As a toy model for the exploration of four dimensional QCD at strong coupling, $\mathcal{N} = 4$ SYM, with its large supersymmetry, conformal invariance, and large number of colors N (that ensures that the dual is just a gravitational theory and not a full string theory) is nearly ideal. By modifying this simple set-up, in particular by breaking supersymmetry and conformal invariance, a lot was learned about QCD itself as, for instance, in the Sakai-Sugimoto model [2]. A crucial part of this development is that the physics of gauge theories at finite temperature shows remarkable universality, which has translated into applications of $\mathcal{N} = 4$ SYM to the high temperature plasmas at RHIC and the ALICE experiment at the LHC (see for example [3] for an extensive review and references).

With the discovery of the pp-wave/BMN correspondence in 2002 [4] came the realization of the importance of operators with large R-charge to a full string theory (and not just supergravity) description of the dual. This, in turn, led to the description of spin chains from string theory [5] and, more generally, to an understanding of the integrable structures on both sides of the correspondence. In a sense, this was the precursor to the application of the gauge/gravity duality to condensed matter physics. More recently, another important example of AdS/CFT appeared, the so-called ABJM model; a three dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons gauge theory with the gauge group $SU(N) \times SU(N)$, dual to string theory on $AdS_4 \times CP^3$. This model can be considered as a prototype for strongly coupled theories in three dimensions, in particular for planar condensed matter systems. For instance, in [6, 7] it was used to study the relativistically invariant quantum critical phase and compressible Fermi surfaces, respectively. These applications of the *AdS/CFT* correspondence to condensed matter hinge on the idea that, if physics in AdS is always holographic, then we can consider simple theories in AdS, which *should* be dual to some strongly coupled conformal field theories (see e.g. [8, 9] for a review). In an overwhelming majority of cases considered, the argument for applying the AdS/CFT duality (and trusting the answers it provided) was *universality*. In other words, the variety of theories usually considered contain a small subset of abelian operators dual to a small number of fields in AdS, usually a gauge field, some scalars and perhaps

some fermions. On the other hand, the relevant condensed matter models one usually wants to describe is usually abelian to begin with. It is not entirely clear then why we can either: i) focus on a small subset of abelian operators of a large N system; or ii) consider an abelian analog of the large N system, which would not have a gravity dual.

A better motivated scenario for such an “AdS/CMT ” correspondence would be if, in a large N field theory with a gravity dual, we could identify a *consistent truncation* of the (in general, nonabelian) field theory to an abelian subset corresponding to the *collective dynamics* of a large number of fields, and the resulting abelian theory would be a relevant condensed matter model. It is toward this end that we explore possible abelian reductions of the ABJM model in this article. Our strategy will be to look to the matrices G^α that characterize the “fuzzy funnel” BPS state of pure ABJM and the “fuzzy sphere” ground state of the massive deformation of ABJM (mABJM) since they correspond to a collective motion of $\mathcal{O}(N)$ out of $\mathcal{O}(N^2)$ degrees of freedom. They will play a central role in our abelianization ansatz. We will then argue that this ansatz furnishes a *consistent truncation* of mABJM and can be used to identify further (phenomenologically) interesting abelianizations. We then show how these find application in condensed matter physics and, finally, we will explore some BPS solutions suggested by the abelian ansätze together with their spacetime interpretation. The main ideas about the abelianization and application to AdS/CMT were outlined in the letter [10], and here we present the full details.

The rest of this paper is organized as follows. In section 2 we explore general abelianization ansätze involving G^α , and identify two important cases of further consistent truncations for this model. In section 3 we study one of them, which, for BPS states, leads to a Toda system that possesses vortex-type solutions with topological charge and finite energy, but with $|\phi| \rightarrow 0$ at both $r \rightarrow 0$ and $r \rightarrow \infty$, which we describe numerically. In section 4 we describe a second case, more relevant for the AdS/CMT motivation above and find a reduction that, depending on certain parameters, gives us either an abelian-Higgs model, or a ϕ^4 (relativistic Landau-Ginzburg) theory. In section 5 we study the relevance of this reduction for condensed matter and AdS/CMT and sketch a simple condensed matter model that reproduces the general features of abelianization. In section 6 we study some BPS solutions suggested by the abelianizations. Finally, in section 7, we provide a possible spacetime interpretation for these solutions in terms of M2-branes on a background spacetime.

2. ABJM, massive ABJM and their Truncations

The ABJM model [11] is obtained as the IR limit of the theory of N coincident

M2-branes moving in $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$. It is a $\mathcal{N} = 6$ supersymmetric $U(N) \times U(N)$ Chern-Simons gauge theory at level $(k, -k)$, with bifundamental scalars C^I and fermions ψ_I , $I = 1, \dots, 4$ in the fundamental of the $SU(4)_R$ symmetry group. The gauge fields are denoted by A_μ and \hat{A}_μ . Its action is given by

$$\begin{aligned}
S = \int d^3x & \left(\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \right. \\
& - \text{Tr} D_\mu C_I^\dagger D^\mu C^I - i \text{Tr} \psi^{I\dagger} \gamma^\mu D_\mu \psi_I + \frac{4\pi^2}{3k^2} \text{Tr} \left(C^I C_I^\dagger C^J C_J^\dagger C^K C_K^\dagger \right. \\
& + C_I^\dagger C^I C_J^\dagger C^J C_K^\dagger C^K + 4C^I C_J^\dagger C^K C_I^\dagger C^J C_K^\dagger - 6C^I C_J^\dagger C^J C_I^\dagger C^K C_K^\dagger \\
& + \frac{2\pi i}{k} \text{Tr} \left(C_I^\dagger C^I \psi^{J\dagger} \psi_J - \psi^{J\dagger} C^I C_I^\dagger \psi_J - 2C_I^\dagger C^J \psi^{I\dagger} \psi_J + 2\psi^{J\dagger} C^I C_J^\dagger \psi_J \right. \\
& \left. \left. + \epsilon^{IJKL} C_I^\dagger \psi_J C_K^\dagger \psi_L - \epsilon_{IJKL} C^I \psi^{J\dagger} C^K \psi^{L\dagger} \right) \right) \quad (2.1)
\end{aligned}$$

where the gauge-covariant derivative is

$$D_\mu C^I = \partial_\mu C^I + iA_\mu C^I - iC_I \hat{A}_\mu. \quad (2.2)$$

The action has a $SU(4) \times U(1)$ R-symmetry associated with the $\mathcal{N} = 6$ supersymmetries.

There is a maximally supersymmetric (*i.e.*, preserving all $\mathcal{N} = 6$) massive deformation of the model with a parameter μ [12, 13], which breaks the R-symmetry down to $SU(2) \times SU(2) \times U(1)_A \times U(1)_B \times \mathbb{Z}_2$ by splitting the scalars as

$$C^I = (Q^\alpha, R^\alpha); \quad \alpha = 1, 2 \quad (2.3)$$

The \mathbb{Z}_2 action swaps the matter fields Q^α and R^α , while the $SU(2)$ factors act individually on the doublets Q^α and R^α respectively and $U(1)_A$ symmetry rotates Q^α with a phase $+1$ and R^α with a phase -1 . The mass deformation, besides giving a mass to the fermions, changes the potential of the theory. The bosonic part of the deformed action can be written as

$$\begin{aligned}
\mathcal{L}_{\text{Bosonic}} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} & \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \right) \\
& - \text{Tr} |D^\mu Q^\alpha|^2 - \text{Tr} |D^\mu R^\alpha|^2 - V \quad (2.4)
\end{aligned}$$

where the potential is

$$V = \text{Tr} (|M^\alpha|^2 + |N^\alpha|^2), \quad (2.5)$$

and where

$$\begin{aligned}
M^\alpha &= \mu Q^\alpha + \frac{2\pi}{k} \left(2Q^{[\alpha} Q_\beta^\dagger Q^{\beta]} + R^\beta R_\beta^\dagger Q^\alpha - Q^\alpha R_\beta^\dagger R^\beta + 2Q^\beta R_\beta^\dagger R^\alpha - 2R^\alpha R_\beta^\dagger Q^\beta \right), \\
N^\alpha &= -\mu R^\alpha + \frac{2\pi}{k} \left(2R^{[\alpha} R_\beta^\dagger R^{\beta]} + Q^\beta Q_\beta^\dagger R^\alpha - R^\alpha Q_\beta^\dagger Q^\beta + 2R^\beta Q_\beta^\dagger Q^\alpha - 2Q^\alpha Q_\beta^\dagger R^\beta \right).
\end{aligned}$$

(2.6)

The equations of motion of the bosonic Lagrangian (2.4) are

$$\begin{aligned} D_\mu D^\mu Q^\alpha &= \frac{\partial V}{\partial Q_\alpha^\dagger}, & D_\mu D^\mu R^\alpha &= \frac{\partial V}{\partial R_\alpha^\dagger}, \\ F_{\mu\nu} &= \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} J^\lambda, & \hat{F}_{\mu\nu} &= \frac{2\pi}{k} \epsilon_{\mu\nu\lambda} \hat{J}^\lambda, \end{aligned} \quad (2.7)$$

where the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ and the two gauge currents J^μ and \hat{J}^μ , given by

$$\begin{aligned} J^\mu &= i (Q^\alpha (D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha) Q_\alpha^\dagger + R^\alpha (D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha) R_\alpha^\dagger), \\ \hat{J}^\mu &= -i (Q_\alpha^\dagger (D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger (D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha), \end{aligned} \quad (2.8)$$

are covariantly conserved so that $\nabla_\mu J^\mu = \nabla_\mu \hat{J}^\mu = 0$. In addition, there are two abelian currents j^μ and \hat{j}^μ corresponding to the global $U(1)_A$ and $U(1)_B$ invariances, given by

$$\begin{aligned} j^\mu &= i \text{Tr} (Q^\alpha (D^\mu Q^\alpha)^\dagger - (D^\mu Q^\alpha) Q_\alpha^\dagger + R^\alpha (D^\mu R^\alpha)^\dagger - (D^\mu R^\alpha) R_\alpha^\dagger), \\ \hat{j}^\mu &= -i \text{Tr} (Q_\alpha^\dagger (D^\mu Q^\alpha) - (D^\mu Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger (D^\mu R^\alpha) - (D^\mu R^\alpha)^\dagger R^\alpha), \end{aligned} \quad (2.9)$$

which are ordinarily conserved *i.e.* $\partial_\mu j^\mu = \partial_\mu \hat{j}^\mu = 0$. By choosing the gauge $A_0 = \hat{A}_0 = 0$, the energy density (Hamiltonian) is given by¹

$$\begin{aligned} H &= \text{Tr} [(D_0 Q^\alpha)^\dagger (D_0 Q^\alpha) + (D_i Q^\alpha)^\dagger (D_i Q^\alpha) \\ &\quad + (D_0 R^\alpha)^\dagger (D_0 R^\alpha) + (D_i R^\alpha)^\dagger (D_i R^\alpha) + V]. \end{aligned} \quad (2.10)$$

Since this is a Chern-Simons theory, the equations of motion must be supplemented with the Gauss law constraints

$$\begin{aligned} F_{12} &= \frac{2\pi i}{k} J^0 = \frac{2\pi i}{k} (Q^\alpha (D^0 Q^\alpha)^\dagger - (D^0 Q^\alpha) Q_\alpha^\dagger + R^\alpha (D^0 R^\alpha)^\dagger - (D^0 R^\alpha) R_\alpha^\dagger), \\ \hat{F}_{12} &= \frac{2\pi i}{k} \hat{J}^0 = -\frac{2\pi i}{k} (Q_\alpha^\dagger (D^0 Q^\alpha) - (D^0 Q^\alpha)^\dagger Q^\alpha + R_\alpha^\dagger (D^0 R^\alpha) - (D^0 R^\alpha)^\dagger R^\alpha). \end{aligned} \quad (2.11)$$

The gauge choice is not as restrictive as it would seem. Choosing (as we do below for our abelianization) A_0 and \hat{A}_0 different from zero produces an extra term in the Hamiltonian of the form $\epsilon^{\mu\nu\lambda} \text{Tr}[A_\mu A_\nu A_\lambda - \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda]$. In the abelian case this vanishes anyway since it is proportional to $\epsilon^{\mu\nu\lambda} a_\mu^{(i)} a_\nu^{(j)} a_\lambda^{(k)}$ and there are only two

¹Note that the terms $A_1 \hat{A}_2 - A_2 \hat{A}_1$ cancel from $p\dot{q} - L$, and the rest of the CS term involve A_0

$a_\mu^{(i)}$'s. So in the abelian case, the Hamiltonian is the same even away from the gauge $A_0 = \hat{A}_0 = 0$. The mass deformed theory has ground states of the fuzzy sphere type given by

$$R^\alpha = cG^\alpha; \quad Q^\alpha = 0 \quad \text{and} \quad Q_\alpha^\dagger = cG^\alpha; \quad R^\alpha = 0 \quad (2.12)$$

where $c \equiv \sqrt{\frac{\mu k}{2\pi}}$ and the matrices G^α , $\alpha = 1, 2$, satisfy the equations [12, 13]

$$G^\alpha = G^\alpha G_\beta^\dagger G^\beta - G^\beta G_\beta^\dagger G^\alpha. \quad (2.13)$$

It was shown in [14, 15] that this solution corresponds to a fuzzy 2-sphere.

An explicit solution of these equations is given by

$$\begin{aligned} (G^1)_{m,n} &= \sqrt{m-1} \, \delta_{m,n}, \\ (G^2)_{m,n} &= \sqrt{(N-m)} \, \delta_{m+1,n}, \\ (G_1^\dagger)_{m,n} &= \sqrt{m-1} \, \delta_{m,n}, \\ (G_2^\dagger)_{m,n} &= \sqrt{(N-n)} \, \delta_{n+1,m}. \end{aligned} \quad (2.14)$$

Clearly, these matrices satisfy $G^1 = G_1^\dagger$ also. In the case of the pure ABJM, there is a *BPS solution* of the fuzzy funnel type with c replaced by

$$c(s) = \sqrt{\frac{k}{4\pi s}}, \quad (2.15)$$

instead, where s is one of the two spatial coordinates of the ABJM model. The matrices G^α are bifundamental under $U(N) \times U(N)$, therefore $G^1 G_1^\dagger$ and $G^2 G_2^\dagger$ are in the adjoint of the first $U(N)$, and $G_1^\dagger G^1$ and $G_2^\dagger G^2$ are in the adjoint of the second.

2.1 An Abelianization Ansatz

Given all these properties of the G^α matrices, it is reasonable to choose the following abelianization ansatz

$$\begin{aligned} A_\mu &= a_\mu^{(2)} G^1 G_1^\dagger + a_\mu^{(1)} G^2 G_2^\dagger, \\ \hat{A}_\mu &= a_\mu^{(2)} G_1^\dagger G^1 + a_\mu^{(1)} G_2^\dagger G^2, \\ Q^\alpha &= \phi_\alpha G^\alpha, \\ R^\alpha &= \chi_\alpha G^\alpha, \end{aligned} \quad (2.16)$$

with no summation over α in the ansatz for Q^α, R^α ; $a_\mu^{(1)}$ and $a_\mu^{(2)}$ real-valued vector fields and ϕ_α, χ_α complex-valued scalar fields.

Since $G^1 G_1^\dagger$ commutes with $G^2 G_2^\dagger$ and $G_1^\dagger G^1$ commutes with $G_2^\dagger G^2$, the gauge fields $a_\mu^{(i)}$ are abelian and the field strengths decompose as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = f_{\mu\nu}^{(2)} G^1 G_1^\dagger + f_{\mu\nu}^{(1)} G^2 G_2^\dagger, \\ \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i[\hat{A}_\mu, \hat{A}_\nu] = f_{\mu\nu}^{(2)} G_1^\dagger G^1 + f_{\mu\nu}^{(1)} G_2^\dagger G^2, \end{aligned} \quad (2.17)$$

with the *abelian* field strengths $f_{\mu\nu}^{(i)} = \partial_\mu a_\nu^{(i)} - \partial_\nu a_\mu^{(i)}$.

With this ansatz, the Chern-Simons term becomes

$$-\frac{k}{4\pi} \frac{N(N-1)}{4} \epsilon^{\mu\nu\lambda} \left(a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)} \right), \quad (2.18)$$

while the covariant derivatives $D_\mu Q^\alpha$ and $D_\mu R^\alpha$ give rise to

$$D_\mu \phi_i = (\partial_\mu - i a_\mu^{(i)}) \phi_i, \quad (2.19)$$

$$D_\mu \chi_i = (\partial_\mu - i a_\mu^{(i)}) \chi_i,$$

and the values for M^α, N^α are given by

$$\begin{aligned} M^1 &= \frac{2\pi}{k} [\phi_1 (c^2 + |\phi_2|^2 - |\chi_2|^2) - 2\chi_1 \bar{\chi}_2 \phi_2] G^1, \\ M^2 &= \frac{2\pi}{k} [\phi_2 (c^2 + |\phi_1|^2 - |\chi_1|^2) - 2\bar{\chi}_1 \chi_2 \phi_1] G^2, \\ N^1 &= \frac{2\pi}{k} [\chi_1 (-c^2 + |\chi_2|^2 - |\phi_2|^2) - 2\phi_1 \bar{\phi}_2 \chi_2] G^1, \\ N^2 &= \frac{2\pi}{k} [\chi_2 (-c^2 + |\chi_1|^2 - |\phi_1|^2) - 2\bar{\phi}_1 \phi_2 \chi_1] G^2 \end{aligned} \quad (2.20)$$

where as before, $c^2 = \mu k / (2\pi)$. Substituting into the potential gives

$$\begin{aligned} V &= \frac{2\pi^2}{k^2} N(N-1) \left[(|\phi_1|^2 + |\chi_1|^2)(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 \right. \\ &\quad + (|\phi_2|^2 + |\chi_2|^2)(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 \\ &\quad \left. + 4|\phi_1|^2 |\phi_2|^2 (|\chi_1|^2 + |\chi_2|^2) + 4|\chi_1|^2 |\chi_2|^2 (|\phi_1|^2 + |\phi_2|^2) \right]. \end{aligned} \quad (2.21)$$

Note that the interchange of χ with ϕ (which changes Q^α with R^α) is equivalent to a change in the sign of c^2 , *i.e.* either a change in the sign of μ , or of k . Putting everything together then gives the final *abelian* effective action

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} (a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)}) + |D_\mu \phi_i|^2 + |D_\mu \chi_i|^2 + U(|\phi_i|, |\chi_i|) \right], \quad (2.22)$$

with a rescaled potential $U \equiv 2V/N(N-1)$. Since the effective theory derives from a Chern-Simons theory, the equations of motion need to be supplemented with the Gauss law constraints which, in our ansatz, reduce to

$$\begin{aligned} f_{12}^{(2)} &= \frac{2\pi i}{k} [\phi_1 (D^0 \phi_1)^\dagger - (D^0 \phi_1) \phi_1^\dagger + \chi_1 (D^0 \chi_1)^\dagger - (D^0 \chi_1) \chi_1^\dagger], \\ f_{12}^{(1)} &= \frac{2\pi i}{k} [\phi_2 (D^0 \phi_2)^\dagger - (D^0 \phi_2) \phi_2^\dagger + \chi_2 (D^0 \chi_2)^\dagger - (D^0 \chi_2) \chi_2^\dagger], \end{aligned} \quad (2.23)$$

We see, however, that these are nothing but the $a_0^{(1)}, a_0^{(2)}$ equations of motion for the action (2.22). As we will need to work away from the $a_0^{(1)} = a_0^{(2)} = 0$ gauge, we don't need to impose them.

2.2 Consistent Truncations

A key point to note about this abelianization ansatz is that it is a *consistent truncation* of the original ABJM theory in the sense that, using the facts that $M^\alpha \propto G^\alpha$, $N^\alpha \propto G^\alpha$, $D_\mu D^\mu(\phi_\alpha G^\alpha) = (D_\mu D^\mu \phi_\alpha)G^\alpha$ and $D_\mu D^\mu(\chi_\alpha G^\alpha) = (D_\mu D^\mu \chi_\alpha)G^\alpha$, the equations of motion that follow from the action (2.22),

$$\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} f_{\mu\nu}^{(1)} = i \left[\bar{\phi}_2 D^\lambda \phi_2 - \phi_2 \overline{D^\lambda \phi_2} + \bar{\chi}_2 D^\lambda \chi_2 - \chi_2 \overline{D^\lambda \chi_2} \right], \quad (2.24)$$

$$\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} f_{\mu\nu}^{(2)} = i \left[\bar{\phi}_1 D^\lambda \phi_1 - \phi_1 \overline{D^\lambda \phi_1} + \bar{\chi}_1 D^\lambda \chi_1 - \chi_1 \overline{D^\lambda \chi_1} \right],$$

and

$$\begin{aligned} & D_\mu D^\mu \phi_1 \\ &= \frac{4\pi^2}{k^2} \left[(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 + 2(|\phi_2|^2 + |\chi_2|^2)(|\phi_1|^2 + |\chi_1|^2 + c^2) + 4|\phi_2|^2 |\chi_2|^2 \right] \phi_1, \\ & D_\mu D^\mu \phi_2 \\ &= \frac{4\pi^2}{k^2} \left[(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 + 2(|\phi_1|^2 + |\chi_1|^2)(|\phi_2|^2 + |\chi_2|^2 + c^2) + 4|\phi_1|^2 |\chi_1|^2 \right] \phi_2, \\ & D_\mu D^\mu \chi_1 \\ &= \frac{4\pi^2}{k^2} \left[(|\chi_2|^2 - |\phi_2|^2 - c^2)^2 + 2(|\phi_2|^2 + |\chi_2|^2)(|\phi_1|^2 + |\chi_1|^2 - c^2) + 4|\phi_2|^2 |\chi_2|^2 \right] \chi_1, \\ & D_\mu D^\mu \chi_2 \\ &= \frac{4\pi^2}{k^2} \left[(|\chi_1|^2 - |\phi_1|^2 - c^2)^2 + 2(|\phi_1|^2 + |\chi_1|^2)(|\phi_2|^2 + |\chi_2|^2 - c^2) + 4|\phi_1|^2 |\chi_1|^2 \right] \chi_2, \end{aligned} \quad (2.25)$$

satisfy the higher original ABJM equations of motion (2.7) and Gauss constraints (2.12).

Since $\text{Tr}[G^1 G_1^\dagger] = \text{Tr}[G_1^\dagger G_1] = \text{Tr}[G^2 G_2^\dagger] = \text{Tr}[G_2^\dagger G_2] = N(N-1)/2$, the energy density (Hamiltonian) is

$$H = \frac{N(N-1)}{2} [|D_0 \phi_i|^2 + |D_0 \chi_i|^2 + |D_a \phi_i|^2 + |D_a \chi_i|^2] + V \quad (2.26)$$

where $a, b = 1, 2$. Note also that away from the gauge $A_0 = \hat{A}_0 = 0$ (which imply that $a_0^{(i)} = 0$), we would, in principle, have a term cubic in the gauge fields in the Hamiltonian. This however vanishes in the abelian case, so the above result is correct

in general. This abelianization, with its four complex scalar fields, is rather general. We will study further reductions of it involving only two scalars. Looking at the scalar equations of motion above we see that putting any two of the scalars to zero is again a consistent truncation.

- A trivial choice turns out to be $\chi_2 = \phi_2 = 0$ (or equivalently $\chi_1 = \phi_1 = 0$), since in that case, the potential reduces to a simple mass term,

$$V = \frac{4\pi^2 c^4}{k^2} (|\phi_1|^2 + |\chi_1|^2) \quad (2.27)$$

while at the same time, the only $a_\mu^{(2)}$ dependence remains in the Chern-Simons term, $\sim \int \epsilon a^{(2)} f^{(1)}$, so its equation of motion is $f_{\mu\nu}^{(1)} = 0$, which means $a_\mu^{(1)}$ is also trivial (pure gauge). So we remain with two massive complex scalar fields coupled to one trivial gauge field (pure gauge, with no kinetic term), an uninteresting model.

- A much more interesting choice is $\phi_1 = \phi_2 = 0$, which will turn out to lead (with some modifications) to the Abelian-Higgs model. Since we will study this separately and extensively in section 4, we will not discuss it further here.
- Finally, setting $\chi_1 = \phi_2 = 0$, and renaming χ_2 to ϕ_2 for simplicity, we get

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \left(a_\mu^{(2)} f_{\nu\lambda}^{(1)} + a_\mu^{(1)} f_{\nu\lambda}^{(2)} \right) + |D_\mu \phi_i|^2 + U(|\phi_i|, |\chi_i|) \right] \quad (2.28)$$

$$V = \frac{2\pi^2}{k^2} N(N-1) [|\phi_1|^2 (|\phi_2|^2 - c^2)^2 + |\phi_2|^2 (|\phi_1|^2 + c^2)^2],$$

and energy density (Hamiltonian)

$$H = \frac{N(N-1)}{2} [|D_0 \phi_i|^2 + |D_a \phi_i|^2] + V. \quad (2.29)$$

We should note that, until now, we have worked only with the *massive* ABJM model, but that we can analyze the *massless* (or pure) ABJM model in a straightforward way by setting $c = 0$. Since c appears only in the potential, we can check that the model with potential (2.21) is symmetric under interchange of $\phi_i \leftrightarrow \chi_i$. For the model with $\chi_1 = \phi_2 = 0$ above, for example, we obtain a purely quartic potential,

$$V = \frac{2\pi^2}{k^2} N(N-1) |\phi_1|^2 |\phi_2|^2 (|\phi_1|^2 + |\phi_2|^2). \quad (2.30)$$

3. New vortex solutions for a Toda system

We now study BPS solutions of the effective model (2.28). Before doing so, it is worth taking a step back, and considering the more general case of the $Q^2 = R^1 = 0$ reduction, with only $Q^1 = Q$ and $R^2 = R$ nonzero, but without the abelianization ansatz. There we can ‘complete squares’ in the Hamiltonian density and write it, in complete analogy to the usual Abelian-Higgs model, as

$$\begin{aligned} \mathcal{H} = & \text{Tr} |D_0 Q - iM|^2 + \text{Tr} |D_0 R + iN|^2 + \text{Tr} |D_- Q|^2 + \text{Tr} |D_+ R|^2 \\ & + i\epsilon^{ab} \partial_a \text{Tr} \left(Q^\dagger (D_b Q) - R^\dagger (D_b R) \right) + \mu j_0, \end{aligned} \quad (3.1)$$

where $\mu j_0 = \mu k / (2\pi) \text{Tr} (F_{12})$ and $D_\pm \equiv D_1 \pm iD_2$. Just as in the Abelian-Higgs model, the term on the second line (with ϵ^{ab}) is zero on the configurations of interest, since

$$\int_V d^2x \epsilon^{ab} \partial_a (\phi^\dagger D_b \phi) = \int_{\partial V_\infty} (\phi^\dagger D_a \phi) dx_a \quad (3.2)$$

and $D_a \phi \rightarrow 0$ at $r \rightarrow \infty$ for $\phi = Q, R$ in order to have finite energy configurations. Moreover, the perfect squares on the first line are minimized by the BPS equations

$$D_- Q = 0; \quad D_+ R = 0; \quad D_0 Q = iM; \quad D_0 R = -iN, \quad (3.3)$$

which leaves just the *topological term*, μj_0 . The BPS equations together with the Gauss law constraints are

$$\begin{aligned} D_- Q &= 0, \\ D_+ R &= 0, \\ D_0 Q &= i\mu Q - \frac{2\pi i}{k} [QR^\dagger R - RR^\dagger Q], \\ D_0 R &= i\mu R + \frac{2\pi i}{k} [RQ^\dagger Q - QQ^\dagger R], \\ F_{12} &= -\frac{4\pi\mu}{k} (QQ^\dagger + RR^\dagger) + \frac{8\pi^2}{k^2} [QR^\dagger RQ^\dagger - RQ^\dagger QR^\dagger], \\ \hat{F}_{12} &= -\frac{4\pi\mu}{k} (Q^\dagger Q + R^\dagger R) + \frac{8\pi^2}{k^2} [R^\dagger QQ^\dagger R - Q^\dagger RR^\dagger Q], \end{aligned} \quad (3.4)$$

where, in the Gauss law constraints, we have already substituted the BPS equations for $D_0 Q, D_0 R$. These equations are more general, and can be used in principle to find nonabelian BPS solutions. In practice, they are still too difficult to solve analytically so from now on we will go back to the abelian case $Q = \phi_1 G^1, R = \phi_2 G^2$. There, the

BPS equations for D_0Q and D_0R become

$$\begin{aligned}(\partial_0 - ia_0^{(1)})\phi_1 &= -\frac{2\pi i}{k}\phi_1 \left[|\phi_2|^2 - \frac{\mu k}{2\pi} \right], \\(\partial_0 - ia_0^{(2)})\phi_2 &= \frac{2\pi i}{k}\phi_2 \left[|\phi_1|^2 + \frac{\mu k}{2\pi} \right].\end{aligned}\tag{3.5}$$

For static configurations (for which $\partial_0\phi_i = 0$) these equations can be solved for $a_0^{(i)}$ as

$$\begin{aligned}a_0^{(1)} &= \frac{2\pi}{k} \left[|\phi_2|^2 - \frac{\mu k}{2\pi} \right], \\a_0^{(2)} &= -\frac{2\pi}{k} \left[|\phi_1|^2 + \frac{\mu k}{2\pi} \right].\end{aligned}\tag{3.6}$$

In other words, the $a_0^{(i)}$ are completely specified by the scalar fields ϕ_i and spatial components of the abelian gauge fields $a_a^{(i)}$, $a = 1, 2$. Consequently, the (temporal) gauge $a_0^{(i)} = 0$ would be inconsistent with the BPS equations. This is different from the Abelian-Higgs model, where the one can set both $a_0 = 0$ and $\partial_0 = 0$, reducing the system to a two dimensional one (for the spatial components). Here, this would be inconsistent with the Gauss law constraint which, for a Chern-Simons gauge field, relates F_{12} to terms with D_0Q and D_0R so that, if F_{12} is nonzero, so too is D_0Q and D_0R . Finally, the Gauss law constraints in the BPS case reduce to

$$\begin{aligned}f_{12}^{(1)} &= -\frac{8\pi^2}{k^2}|\phi_2|^2 \left(|\phi_1|^2 + \frac{\mu k}{2\pi} \right), \\f_{12}^{(2)} &= \frac{8\pi^2}{k^2}|\phi_1|^2 \left(|\phi_2|^2 - \frac{\mu k}{2\pi} \right).\end{aligned}\tag{3.7}$$

In order to facilitate the rest of the analysis of the BPS system, it will prove useful to complexify the (x_1, x_2) -plane and write $z = x^1 + ix^2$. As usual, this induces a complexification of the derivatives as well as the gauge fields as,

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad a^{(i)} \equiv a_z^{(i)} = \frac{1}{2}(a_1^{(i)} - ia_2^{(i)}),$$

together with their complex conjugates. This, in turn, implies that $f_{12}^{(i)} = -2i[\partial\bar{a}^{(i)} - \bar{\partial}a^{(i)}]$, so that the BPS equations $D_-Q = D_+R = 0$ become simply

$$\partial\phi_1 - ia^{(1)}\phi_1 = 0, \quad \bar{\partial}\phi_2 - i\bar{a}^{(2)}\phi_2 = 0.\tag{3.8}$$

Equations (3.8) together with the Gauss law constraints constitute a complete set which, as we argue below, possess at least one simple set of finite energy, spatially

localized solutions of the vortex type *i.e. isolated zeros of the (complex) scalar fields with nonvanishing winding number*. The analysis follows the same general logic as for the Nielsen-Olesen vortex and we start by writing

$$\phi_1 = |\phi_1|e^{i\theta_1}; \quad \phi_2 = |\phi_2|e^{i\theta_2} \quad (3.9)$$

and then (3.8) become (after taking derivatives and making the combinations $f_{12}^{(i)}$)

$$\begin{aligned} f_{12}^{(1)} &= 2\partial\bar{\partial}\ln|\phi_1|^2 - 2i(\partial\bar{\partial} - \bar{\partial}\partial)\theta_1 = \frac{1}{2}\Delta\ln|\phi_1|^2 + \epsilon^{ab}\partial_a\partial_b\theta_1, \\ -f_{12}^{(2)} &= 2\partial\bar{\partial}\ln|\phi_2|^2 + 2i(\partial\bar{\partial} - \bar{\partial}\partial)\theta_2 = \frac{1}{2}\Delta\ln|\phi_2|^2 - \epsilon^{ab}\partial_a\partial_b\theta_2. \end{aligned} \quad (3.10)$$

But since, if α is the polar angle in the 1, 2 plane, $\epsilon^{ab}\partial_a\partial_b\alpha = 2\pi\delta^2(x)$ (as can be checked by integrating over a circle of vanishingly small radius), we may take the ansatz

$$\theta_1 = -N_1\alpha, \quad \theta_2 = N_2\alpha, \quad (3.11)$$

which leads to

$$\begin{aligned} f_{12}^{(1)} &= \frac{1}{2}\Delta\ln|\phi_1|^2 - 2\pi N_1\delta^2(x), \\ -f_{12}^{(2)} &= \frac{1}{2}\Delta\ln|\phi_2|^2 - 2\pi N_2\delta^2(x). \end{aligned} \quad (3.12)$$

Finally, on substituting the Gauss law constraints, we obtain a continuous Toda system with delta functions sources,

$$\begin{aligned} \Delta\ln|\phi_1|^2 &= -\left(\frac{4\pi}{k}\right)^2|\phi_2|^2\left(|\phi_1|^2 + \frac{\mu k}{2\pi}\right) + 4\pi N_1\delta^2(x), \\ \Delta\ln|\phi_2|^2 &= -\left(\frac{4\pi}{k}\right)^2|\phi_1|^2\left(|\phi_2|^2 - \frac{\mu k}{2\pi}\right) + 4\pi N_2\delta^2(x), \end{aligned} \quad (3.13)$$

whose solutions we proceed to analyze.

3.1 Asymptotic Analysis of the Toda System

As in the case of the (much simpler) Nielsen-Olesen vortex, the Toda system of equations does not, as far as we are aware, exhibit any closed form analytic solution. Consequently, here too must we resort to topological, asymptotic and numerical analyses to tease out finite energy solutions from it. The argument, fortunately, goes

through in much the same way as for the Neilsen-Olesen case: the topological term in the energy

$$\int_{\mathbb{R}^2} dxdy \mu j_0 = \int_{\mathbb{R}^2} dxdy \frac{\mu k}{2\pi} \text{Tr}(F_{12}) = \frac{\mu k}{2\pi} \frac{N(N-1)}{2} \int_{\mathbb{R}^2} dxdy (f_{12}^1 + f_{12}^2) ,$$

is quantized as usual, since for an abelian gauge field

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12} dxdy = \frac{1}{2\pi} \oint_C A_\alpha dl = \frac{1}{2\pi} \int_0^{2\pi} A_\alpha d\alpha ,$$

and $D_a \phi \rightarrow 0$ at $r \rightarrow \infty$, with $\phi = |\phi|e^{i\theta}$, $D_\mu = \partial_\mu - ia_\mu$ and $\partial_\mu \ln |\phi| \rightarrow 0$, means that $\partial_a \theta^\infty - A_\alpha^\infty = 0$ and consequently

$$\int_{\mathbb{R}^2} F_{12} dxdy = 2\pi \tilde{N} .$$

The corresponding statement for our system (3.8) is that, at infinity

$$\partial_\alpha \theta_i^\infty = a_\alpha^{(i)\infty} ,$$

which gives the energy of the BPS state as

$$E(N_1, N_2) = \mu k \frac{N(N-1)}{2} (N_1 + N_2) . \quad (3.14)$$

We are now in a position to look at the Toda equations in the asymptotic regions. To obtain the $r \rightarrow 0$ behaviour, we integrate each of them over a very small disk of radius $R \rightarrow 0$, and find that

$$\int dxdy \vec{\nabla} \cdot \vec{\nabla} \ln |\phi_i| = 2\pi N_i , \quad i = 1, 2 ,$$

which, after using Stokes' theorem, gives

$$R \frac{d}{dr} \ln |\phi_i|_{r=R} = N_i .$$

This expression is easily integrated to show that as $r \rightarrow 0$ each of the scalars exhibits the power law behaviour,

$$|\phi_i| \sim A_i r^{N_i} . \quad (3.15)$$

This is in accordance with the usual argument says that the only possibility for the vortices with $\phi = |\phi|e^{iN\alpha}$ is to have $|\phi| \rightarrow 0$ at $r \rightarrow 0$ in order that the phase is well defined at $r = 0$. In fact we can do better and refine the conditions at $r \rightarrow 0$ by using the equations of motion away from $r = 0$. Taking as an ansatz for the scalars

$$|\phi_1|^2 = A_1 r^{2N_1} (1 + B_1 r^p) , \quad (3.16)$$

$$|\phi_2|^2 = A_2 r^{2N_2} (1 + B_2 r^q) ,$$

and substituting into the Toda equations, we find

$$\begin{aligned} q &= 2N_1 + 2, \quad p = 2N_2 + 2, \\ B_1 &= \frac{8\pi\mu}{kp^2} A_2, \\ B_2 &= -\frac{8\pi\mu}{kq^2} A_1. \end{aligned} \tag{3.17}$$

The constants A_i are only determined from the full numerical solution.

At $r \rightarrow \infty$, we can first check that neither a constant, nor a decaying exponential, nor a power law that blows up, $\phi \sim r^p$ works for either of the two fields. This leaves a decaying power law as the only plausible behaviour for either of the two scalar fields. In order to use the equations above, we must consider also the first subleading terms, *i.e.* we substitute

$$\begin{aligned} |\phi_1|^2 &= \frac{\bar{A}_1}{r^m} \left(1 + \frac{\bar{B}_1}{r^p} \right), \\ |\phi_2|^2 &= \frac{\bar{A}_2}{r^n} \left(1 + \frac{\bar{B}_2}{r^q} \right), \end{aligned} \tag{3.18}$$

into the equations above, to find

$$\begin{aligned} m &= q + 2, \quad n = p + 2, \\ \bar{B}_1 &= -\left(\frac{8\pi\mu}{p^2 k} \right) \bar{A}_2, \\ \bar{B}_2 &= \left(\frac{8\pi\mu}{q^2 k} \right) \bar{A}_1. \end{aligned} \tag{3.19}$$

Since $p, q \in \mathbb{N}_*$, $m, n = 3, 4, 5, \dots$. Again, the constants \bar{A}_1, \bar{A}_2 , as well as m, n are determined from the full numerical solutions.

3.2 Numerical Analysis of the Toda System

To determine the various parameters of the vortex-like solutions described above, we need to solve the Toda system numerically. As in the asymptotic analysis above, our numerical solution follows the general logic of the Abelian-Higgs model. Specifically, we will use a modified *two-parameter shooting method* to numerically solve the two-point boundary value problem described by the coupled Toda equations. To facilitate the implementation of the shooting algorithm, we first rewrite the equations as a four-dimensional (non-autonomous) dynamical system. To this end, we first non-

dimensionalize the system by rescaling our variables and defining

$$\begin{aligned} g &\equiv \frac{2\pi}{\mu k} |\phi_1|^2, \\ f &\equiv \frac{2\pi}{\mu k} |\phi_2|^2, \\ R &\equiv \frac{r}{2\mu}, \end{aligned} \tag{3.20}$$

with r taken to be the radial coordinate on the plane. Substituting into the system (3.14) and assuming that the solitons that we are looking for are rotationally symmetric on the plane (so that the two-dimensional Laplacian is $\Delta = \frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \right)$), we find that eqs.(3.14) reduce to

$$\begin{aligned} \frac{1}{R} \frac{d}{dR} \left(\frac{R}{f} \frac{df}{dR} \right) &= -g(f-1), \\ \frac{1}{R} \frac{d}{dR} \left(\frac{R}{g} \frac{dg}{dR} \right) &= -f(g+1). \end{aligned} \tag{3.21}$$

Finally, we reduce the order of the system by one by making the additional definitions $h \equiv \frac{df}{dR}$ and $j \equiv \frac{dg}{dR}$ so that (denoting by a ' derivatives with respect to the dimensionless radial variable R)

$$\begin{aligned} f' &= h, \\ h' &= \frac{h^2}{f} - \frac{h}{r} - gf(f-1), \\ g' &= j, \\ j' &= \frac{j^2}{g} - \frac{j}{r} - gf(g+1). \end{aligned} \tag{3.22}$$

Before directly integrating this four-dimensional non-autonomous dynamical system, it will be instructive to extract some qualitative information from it. There are two (physical) fixed points at $(f, h, g, j) = (0, 0, 0, 0)$ and $(1, 0, 0, 0)$. A linearization of the system near the former, shows that the origin is a saddle. Solutions of the kind that carry nonvanishing winding number and conform to the asymptotic boundary conditions $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = 0$ correspond to *homoclinic* orbits² that begin and end at $(0, 0, 0, 0)$ and that encircle the fixed point at $(1, 0, 0, 0)$ (see Fig.1).

Our numerical integration of the system is based on a two-parameter shooting algorithm that converts the nonlinear dynamical system above into a nonlinear parameter estimation problem. The parameters in question are precisely the undetermined

²This should be compared to the standard ANO vortices of the Abelian-Higgs model which correspond to *heteroclinic* orbits interpolating between the two fixed points of the associated dynamical system.

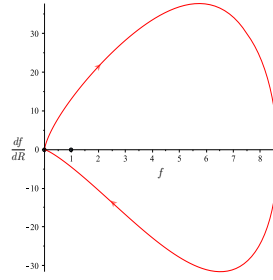


Figure 1: The (f, f') -subspace of the full phase space of the Toda system

constants A_1 and A_2 above and these are chosen at $R = 0$ so that the *constraint* $f(\infty) = g(\infty) = 0$ is met. In practice, the constraints at $R = \infty$ are a problem, but our asymptotic analysis above can be extended to show that solutions at $R \approx 10$ are quite safely in the far field for both f and g . Some results of our numerical integration are presented in Figures 2 and 3.

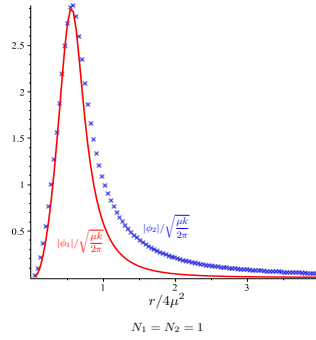


Figure 2: The $N_1 = N_2 = 1$ soliton profiles. Optimization of the shooting parameters yeild $A_1 = 30.00, A_2 = 30.05$

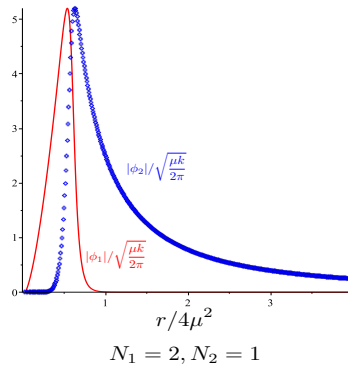


Figure 3: The $N_1 = 2, N_2 = 1$ soliton profiles. Optimization of the shooting parameters yeild $A_1 = 50.00, A_2 = 100.00$

We also obtain from the numerics that the power law at infinity is $|\phi_1|^2 \propto 1/r^3$, $|\phi_2|^2 \propto 1/r^2$, i.e. $m = 3, n = 2$.

As a final point, we note that in the massless ABJM case, at $\mu = 0$, these vortices vanish since their energy is proportional to μ . This agrees well with known facts about the solitonic spectrum of pure ABJM [16].

4. The Abelian-Higgs model from ABJM

We now look to embed the Abelian-Higgs model in ABJM, as a truncation of our general abelianization ansatz. To find the truncation we look at the multi-vortex solution we found previously in [17] for the $N = 2$ case, i.e. $U(2) \times U(2)$ ABJM. There, not only was the ansatz written in a manner similar to the multi-vortices of the conventional Abelian-Higgs model, but the action on the moduli space of vortices was also found to be the same. In retrospect, this was really a telling signal that we were actually embedding the Abelian-Higgs model into ABJM. For the reader unfamiliar with [17], we recall that the static multivortex solution there was given by

$$\begin{aligned} C^1 &= \sqrt{\frac{k\mu}{\pi}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{\frac{k\mu}{\pi}} G^1, \\ C^2 &= \sqrt{\frac{k\mu}{\pi}} e^{-\psi/2} H_0(z) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sqrt{\frac{k\mu}{\pi}} e^{-\psi/2} H_0(z) G^2; \quad C^3 = C^4 = 0, \\ A_0 &= \frac{1}{\mu} \begin{pmatrix} \partial \bar{\partial} \psi & 0 \\ 0 & 0 \end{pmatrix}; \quad \hat{A}_0 = \frac{1}{\mu} \begin{pmatrix} 0 & 0 \\ 0 & \partial \bar{\partial} \psi \end{pmatrix}; \quad A_{\bar{z}} = \hat{A}_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{2} \bar{\partial} \psi \end{pmatrix}, \end{aligned} \quad (4.1)$$

where $H_0(z) = \prod_{i=1}^n (z - z_i)$ is an arbitrary polynomial and the real function $\psi(z)$ is determined through the equation

$$\partial \bar{\partial} \psi = \mu^2 (1 - e^{-\psi} |H_0(z)|^2) \quad (4.2)$$

with boundary conditions at $|z| \rightarrow \infty$ requiring $\psi \rightarrow \log |H_0(z)|^2$. As usual, z_i with $i = 1 \dots n$, denotes the positions of the n vortices. Treating each of these position variables as (adiabatic) functions of time, $z_i(t)$, produces the first order solution

$$\begin{aligned} C^I &= 0, \\ A_0^{(1)} &= \hat{A}_0^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{i}{2} (\dot{z}^i \partial_i - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \psi \end{pmatrix}, \\ A_{\bar{z}}^{(1)} &= \begin{pmatrix} \frac{1}{2\mu} \dot{z}^i \partial_i \partial \psi & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{A}_{\bar{z}}^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2\mu} \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}} \partial \psi \end{pmatrix}, \end{aligned} \quad (4.3)$$

on the moduli space of the vortices. On the other hand, when $N = 2$, our general abelianization ansatz gives

$$A_\mu = a_\mu^{(1)} G^1 G_1^\dagger + a_\mu^{(2)} G^2 G_2^\dagger = \begin{pmatrix} a_\mu^{(1)} & 0 \\ 0 & a_\mu^{(2)} \end{pmatrix}, \quad (4.4)$$

$$\hat{A}_\mu = a_\mu^{(1)} G_1^\dagger G^1 + a_\mu^{(2)} G_2^\dagger G^2 = \begin{pmatrix} 0 & 0 \\ 0 & a_\mu^{(1)} + a_\mu^{(2)} \end{pmatrix}.$$

Comparing with the solution above (and also denoting now the first order solution for the abelian fields with a tilde to avoid confusion with the indices (1) and (2) on the a 's) we find

$$\begin{aligned} a_{\bar{z}}^{(2)} &= \frac{i}{2} \bar{\partial} \psi, & a_0^{(1)} &= \frac{1}{\mu} \partial \bar{\partial} \psi, & a_a^{(1)} &= a_0^{(2)} = 0, \\ \tilde{a}_{\bar{z}}^{(1)} &= \frac{1}{2\mu} \dot{z}^{\bar{i}} \partial_{\bar{i}} \partial \psi, \\ \tilde{a}_0^{(2)} &= -\frac{i}{2} (\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}}) \psi. \end{aligned} \quad (4.5)$$

Note that from (4.5), by taking complex conjugate and then sums and differences, we get

$$\begin{aligned} \tilde{a}_1^{(1)} &= \frac{1}{2\mu} \left[\left(\dot{z}^{\bar{i}} \partial_{\bar{i}} + \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}} \right) \partial_1 - i \left(\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}} \right) \partial_2 \right] \psi, \\ \tilde{a}_2^{(1)} &= -\frac{i}{2\mu} \left[\left(\dot{z}^{\bar{i}} \partial_{\bar{i}} - \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}} \right) \partial_1 - i \left(\dot{z}^{\bar{i}} \partial_{\bar{i}} + \dot{\bar{z}}^{\bar{i}} \bar{\partial}_{\bar{i}} \right) \partial_2 \right] \psi. \end{aligned} \quad (4.6)$$

This can be written more compactly, by using the fact that ψ is a real-valued field, as

$$\tilde{a}_i^{(1)} = \frac{i}{2\mu} \epsilon_{ij} \left(\dot{z}^{\bar{k}} \partial_{\bar{k}} - \dot{\bar{z}}^{\bar{k}} \bar{\partial}_{\bar{k}} \right) \partial_j \psi. \quad (4.7)$$

In view of the above solution, and assuming that the same relation to our abelianization holds at all N , we can now identify the truncation ansatz needed to obtain the abelian-Higgs model as

$$\phi_1 = \phi_2 = 0, \chi_1 = b = \text{constant}, \quad (4.8)$$

which gives

$$\begin{aligned} D_\mu \phi_1 &= D_\mu \phi_2 = 0, \\ D_\mu \chi_1 &= -i a_\mu^{(1)} b \\ D_\mu \chi_2 &= (D_\mu - i a_\mu^{(2)}) \chi_2 \end{aligned} \quad (4.9)$$

and the potential

$$\begin{aligned} V &= \frac{2\pi^2}{k^2} N(N-1) [|b|^2 (|\chi_2|^2 - c^2)^2 + |\chi_2|^2 (|b|^2 - |c|^2)^2], \\ &= \frac{2\pi^2}{k^2} N(N-1) [|b|^2 |\chi_2|^4 + |\chi_2|^2 (-4|b|^2 c^2 + |b|^4 + c^4) + c^4 |b|^2]. \end{aligned} \quad (4.10)$$

We can easily arrange for the coefficient of the $|\chi|^2$ term to be negative, as is required for the mexican hat potential of the abelian-Higgs model, by choosing for instance

$$|b| = |c| \Rightarrow \mu = \frac{2\pi|b|^2}{k}. \quad (4.11)$$

The action is then

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu^{(1)} f_{\nu\lambda}^{(2)} + (a_\mu^{(1)})^2 |b|^2 + |D_\mu \chi_2|^2 + V \right], \quad (4.12)$$

with the auxiliary field $a_\mu^{(1)}$. As usual it can be eliminated through its equation of motion

$$a_\mu^{(1)} = -\frac{k}{4\pi|b|^2} \epsilon^{\mu\nu\lambda} f_{\nu\lambda}^{(2)}. \quad (4.13)$$

so that

$$S = -\frac{N(N-1)}{2} \int d^3x \left[\frac{k^2}{8\pi^2|b|^2} (f_{\mu\nu}^{(2)})^2 + |D_\mu \chi_2|^2 + V \right], \quad (4.14)$$

which is nothing but the action of the abelian-Higgs model.

Of course, we still need to check the consistency of the truncation, i.e. to check that the equations of motion of the full abelianization ansatz in section 2 are satisfied. We have fixed ϕ_1, ϕ_2 to zero and χ_1 to b , so it is these three equations of motion that we need to check. As before, the choice $\phi_1 = \phi_2 = 0$ is a consistent truncation. The equation for χ_1 reduces, in the Lorentz gauge $\partial^\mu a_\mu^{(2)} = 0$, and using (4.13), to

$$(a_\mu^{(1)})^2 b = b \frac{\partial V}{\partial |b|^2} \quad (4.15)$$

which is just the equation of motion we would obtain for the parameter b by varying in the abelian-Higgs action (4.14). We find this somewhat puzzling, since it means that the *constant parameter* $|b|$ has to be effectively treated like a field in the abelian-Higgs action, giving its own equation of motion.

It remains now to check that our multivortex solution satisfies the condition (4.13), since it certainly matched our ansatz *before* we imposed the equation of motion for $a_\mu^{(1)}$. The equations (4.13) reduce for the zeroth order solution and the first order

solution respectively, to

$$a_0^{(1)} = \frac{k}{4\pi|b|^2} \epsilon^{ij} \partial_i a_j^{(2)}$$

$$\tilde{a}_i^{(1)} = -\frac{k}{4\pi|b|^2} \epsilon^{ij} \partial_j \tilde{a}_0^{(2)}$$
(4.16)

We can check the first equation, since $a_{\bar{z}}^{(2)} = \frac{i}{2} \bar{\partial} \psi$, which written in real components reads $\tilde{a}_i^{(2)} = -\frac{1}{2} \epsilon_{ij} \partial_j \psi$. Then the zeroth order equation is satisfied if

$$\mu = \frac{2\pi|b|^2}{k}$$
(4.17)

while the first order equation is satisfied when

$$\tilde{a}_i^{(1)} = \frac{i}{2\mu} \epsilon_{ij} (z^{\bar{k}} \partial_{\bar{k}} - \bar{z}^{\bar{k}} \bar{\partial}_{\bar{k}}) \partial_j \psi$$
(4.18)

as expected. The appearance of the abelian-Higgs model above is somewhat non-standard but can easily be put into canonical form by appropriately normalizing the fields as

$$a^{(2)} = \frac{2\pi b}{Nk} \tilde{a}^{(2)}, \quad \chi_2 = \frac{\tilde{\chi}_2}{N},$$

to obtain

$$S = \int d^3x \left[-\frac{1}{4} \left(\tilde{f}_{\mu\nu}^{(2)} \right)^2 - |D_\mu \tilde{\chi}_2|^2 - V \right]$$
(4.19)

where now $D_\mu = \partial_\mu - ig \tilde{a}_\mu^{(2)}$ and $g = \frac{2\pi|b|}{Nk}$. In terms of the canonical fields and coupling, the potential

$$V = \frac{g^2}{2} \left[|\tilde{\chi}_2|^4 + \frac{\mu^2 k^2 N^4}{4\pi^2} + |\tilde{\chi}_2|^2 N^2 \left(-\frac{4\mu k}{2\pi} + |b|^2 + \frac{\mu^2 k^2}{4\pi^2 |b|^2} \right) \right].$$
(4.20)

As previously alluded to, the potential has a range of values of $|b|$ for which it is spontaneously breaking (has negative mass squared). The central value of this domain is $|b| = c$, and for this value of $|b|$, we find

$$V = \frac{g^2}{2} \left[|\tilde{\chi}_2|^2 - \frac{\mu k N^2}{2\pi} \right].$$
(4.21)

Moreover, for this value of $|b|$, the equation of motion for $|b|^2$ (the extra constraint on our abelian-Higgs model), becomes

$$[|\phi_2|^2 - c^2]^2 = \frac{1}{2} \left[\frac{k}{2\pi\mu} f_{\mu\nu}^{(2)} \right]^2.$$
(4.22)

On the other hand, equating the kinetic (Maxwell) term for $f_{\mu\nu}^{(2)}$ with the potential term, V , gives exactly the same equation. Further, taking the square root of this equation, and imposing that $f_{0i} = 0$, we find

$$\frac{1}{Ng}f_{12}^{(2)} = \pm[|\tilde{\chi}_2| - N^2c^2], \quad (4.23)$$

which is part of the abelian-Higgs BPS condition. In other words, the extra condition is satisfied on BPS solutions of the abelian-Higgs model with $f_{0i} = 0$, and in particular for vortices.

This completes our demonstration that (i) the abelian Higgs model can be obtained from the abelianization of the ABJM model and (ii) that both the classical (zeroth order) and the first order (in the moduli space approximation) multivortex solutions of the latter at $N = 2$, are encoded in this model. Since this is a *bone fide* embedding of the abelian-Higgs model, we say more even. For instance, it is natural that we obtain the same fluctuation action for vortex scattering as in the abelian Higgs case. It also means that we can now immediately write down the multivortex solution at general N , with the guarantee that we will recover the same fluctuation action for vortex scattering as in the abelian Higgs case. To be concrete, the multivortex solution at general N in ABJM

$$\begin{aligned} R^1 &= \sqrt{\frac{k\mu}{\pi}}G^1, \quad R^2 = \sqrt{\frac{k\mu}{\pi}}e^{-\psi/2}H_0(z)G^2, \quad Q^1 = Q^2 = 0, \\ A_0 &= \frac{1}{\mu}\partial\bar{\partial}\psi G^1G_1^\dagger, \quad \hat{A}_0 = \frac{1}{\mu}\partial\bar{\partial}\psi G_1^\dagger G^1, \\ A_{\bar{z}} &= \frac{i}{2}\bar{\partial}\psi G^2G_2^\dagger, \quad \hat{A}_{\bar{z}} = \frac{i}{2}\bar{\partial}\psi G_2^\dagger G^2, \end{aligned} \quad (4.24)$$

produces an effective Lagrangian on the moduli space

$$\begin{aligned} L_{eff} &= \frac{N(N-1)}{2} \frac{k\mu}{\pi} \int d^2x \left[-\partial\bar{\partial}\psi + \frac{1}{2}\dot{z}^i\dot{z}^j \left(\partial_i\bar{\partial}_j\psi + \frac{1}{\mu^2}(\partial_i\bar{\partial}\psi\bar{\partial}_j\partial\psi - \partial\bar{\partial}\psi\partial_i\bar{\partial}_j\psi) \right) \right], \\ &\simeq \frac{N(N-1)}{2} \left[-k\mu n + \sum_{i=1}^n \frac{k\mu}{2}|\dot{z}^i|^2 - k\mu q \sum_{i>j} K_0(2\mu|z^i - z^j|)|\dot{z}^i - \dot{z}^j|^2 \right], \end{aligned} \quad (4.25)$$

with $q \simeq 1.71$. To close this discussion on vortices of the abelian-Higgs model and their embedding into the ABJM model, we mention briefly that in the case of the massless ABJM model, with $c = \mu = 0$, we obtain a non-symmetry breaking potential,

$$V = \frac{2\pi^2}{k^2}N(N-1) \left[|b|^2|\chi_2|^4 + |b|^4|\chi_2|^2 \right], \quad (4.26)$$

which is just a massive gauged ϕ^4 model.

5. Towards a string construction of AdS/CMT

At this point, let's stop and consider what it is that we have achieved. Stripping away all the bells and whistles, essentially our truncation has produced a (2+1)-dimensional scalar field theory with potential

$$V = \frac{2\pi^2}{k^2} N(N-1) [|b|^2 |\phi|^4 + |\phi|^2 ((|b|^2 - c^2)^2 - 2c^2 |b|^2) + c^4 |b|^2] . \quad (5.1)$$

It is not too difficult to see that it is just a *Landau-Ginzburg model* in which, at fixed $|b|^2$, $c^2 \propto \mu$ acts as a coupling that takes us from a $|\phi|^4$ theory (the insulator phase) to an abelian-Higgs theory (the superconducting phase). In this sense, the parameters $|b|^2$ and c^2 control the coupling g and critical coupling g_c of the Landau-Ginzburg model. More precisely, we identify the combinations $(|b|^2 - c^2)^2$ as g and $2c^2 |b|^2$ as g_c respectively. In this light, it makes sense then to think of this abelianization as *a realization of the recently proposed AdS/CMT correspondence*. To see why our construction is markedly different from any of its pre-cursors, we recall the general ideas involved. Usually, in an AdS/CMT construction, one assumes some theory in an AdS background, usually involving gravity, a gauge field A_μ , maybe a complex (charged) scalar ϕ and some fermions ψ_i . It is then argued that this theory should be dual to some large N conformal field theory with a global current J_μ dual to the gauge field A_μ , and some other operators (in principle) dual to the other fields. It is then argued that relevant physics in AdS corresponds to some behaviour of the operators in the field theory which simulates the relevant physics, like superconductivity [18] for example, to be studied. Sometimes the AdS theory is obtained as a consistent truncation of some known AdS/CFT duality (for which there is a heuristic derivation involving a decoupling limit of some brane constructions), so that the field theory contains a small subset of operators that could possibly give the desired physics [8, 9].

However, even in these cases, it is not obvious how to directly relate the set of operators in the given CFT to the condensed matter system of interest, and usually one has to invoke some sort of universality argument. In other words, if the physics of the selected set of operators in the large N CFT describes the correct physics for the condensed matter system, then perhaps the physics is general enough to appear in many different systems, and we can try to apply our seemingly unrelated field theory to the condensed matter system of interest. While we certainly appreciate the logic of this argument, we find it less than satisfactory for a number of reasons. Primary among these is that it is not at all clear why can we choose only a very small number of operators in the large N CFT and concentrate on their physics. Secondly, if we try to write down a gravity dual of an abelian theory having this small number of nontrivial operators, we would fail, since the absence of the large N would mean that we could not focus on the supergravity limit in the dual.

However, we can now do better. We have found a *consistent truncation* of the large N CFT, for which there is a well-defined duality, and not just a truncation of the gravity theory. That means that this set of fields is a well defined subset at the *quantum level*³ corresponding to the collective motion of the nonabelian fields in the large N case and involving $\mathcal{O}(N)$ out of the $\mathcal{O}(N^2)$ fields of ABJM, via the nontrivial matrices G^α (which have $\mathcal{O}(N)$ nonzero elements). It is not just a simple restriction to $N = 1$ of the ABJM model, which would imply losing the supergravity limit in the dual. Therefore this abelianization still maps to a purely gravitational theory, and not a full string theory as for generic abelian theories.

In our case, there already exists a well defined gravity dual of the field theory. In the case of massless ABJM, that theory corresponds to M2-branes moving in the space $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$, and the gravity dual (*i.e.* the near-horizon limit of the backreacted background) is $AdS_4 \times \mathbb{CP}^3$. In the case of the massive ABJM, the theory corresponds to M2-branes moving in a space defined in [17, 19] with the gravity dual described in [20, 17]. Of course, we still would need to understand to what the truncation to $\langle \chi_1 \rangle = b$ and $\chi_2 \neq 0$ corresponds in this gravity dual in order to complete the picture, but we leave this for further work.

Actually, as it turns out, the theory we obtain in the abelianization is also the relevant effective theory for a CMT construction. Indeed, as reviewed for instance in [21], starting from the Hubbard model for spinless bosons hopping on a lattice of sites i with short range repulsive interactions,

$$H_b = -w \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + b_j^\dagger b_i \right) + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu \sum_i n_i, \quad (5.2)$$

where $n_i = b_i^\dagger b_i$ and w is the hopping matrix between nearest-neighbour sites, one obtains the relativistic Landau-Ginzburg theory

$$S = \int d^3x \left(-|\partial_t \phi|^2 + v^2 |\vec{\nabla} \phi|^2 + (g - g_c) |\phi|^2 + u |\phi|^4 \right). \quad (5.3)$$

The effective field ϕ is obtained as follows. The ground state contains an equal number of bosons at each site, with the creation operators a_i^\dagger producing extra particles at each site, and creation operators h_i^\dagger that produce extra “holes” at each site; “antiparticles” in the QFT picture. Then, as is usual in field theory, $\phi_i \sim \alpha_i a_i + \beta_i h_i^\dagger$ is a discretized version of the complex field describing both particles and antiparticles, where α_i, β_i are wavefunctions for the modes.

For $g < g_c$ we have an abelian-Higgs system, *i.e.* a superconducting phase, while for $g > g_c$ we have an insulator phase. The marginal case $g = g_c$ is a conformal

³We can consistently put the other fields to zero even at the quantum level.

field theory. The systems described by the above model also have a quantum critical phase which opens up at nonzero temperature for a T -dependent window around $g = g_c$. This quantum critical phase is strongly coupled and very hard to describe using conventional condensed matter methods, which makes it an excellent choice for a holographic description. In [6] it was shown that by considering a gauge field in the gravity dual of ABJM and introducing a coupling for it to the Weyl curvature, $\gamma \int C_{abcd} F^{ab} F^{cd}$ one obtains a conductivity $\sigma(\omega)$ consistent with the quantum critical phase, and from which it was concluded that ABJM is a good primer for these systems, though the precise reason for the match was not obvious.

While the bosonic Hubbard model leads, in the continuum limit to the action (5.3), the model itself is a drastic simplification, of a condensed matter system. The model has been used to describe the quantum critical phase of (bosonic) ^{87}Rb cold atoms on an optical lattice, but the description is believed to hold more generally for the quantum critical phase. For instance, high T_c superconductors have a “strange metal” phase that is believed to be of the same quantum critical type. We can consider a solid with free electrons (fermions, perhaps several per atom) that could hop between fixed atoms, and unlike the simple Hubbard model, we also have in principle interactions that are not restricted to nearest neighbours. One could, for instance, generate bosons ϕ_{ij} (having the role of the bosons b_i of the Hubbard model) by coupling fermions at two sites i and j . By an abuse of notation we will call by the same ϕ_{ij} the field obtained by multiplying the corresponding “particle creation” operator with a wavefunction, and adding a corresponding “hole” part.

In fact, we can sketch a simple model for the condensed matter system above that generates the same qualitative picture as the abelianization of the ABJM model. Consider spinless bosons ϕ_{ij} generated by coupling fermions of opposite spins (Cooper pairs) at sites i and j with a maximum distance between sites $|i - j| \leq N$, i.e. $\bar{\psi}_i \psi_j$. The resulting ϕ_{ij} can be described by a field $\phi_{i'}^{ab}$, with $a, b = 1, \dots, N$. Since we are in two spatial dimensions, every site has $\mathcal{O}(N^2)$ neighbours a distance $\leq N$ away. Now take the point i' at which the effective field, ϕ_{ij} , lives to be midpoint of the line between i and j , and a, b to correspond to sites j in the x and y directions away from i' (so that, if i' and j are fixed, so is i). Consider that the normalized wavefunctions for the field $\phi_{i'}^{ab}$ give probabilities for existence of the pairing as $\propto |\phi_{i'}^{ab}|^2$ for a pair (ab) . In this case, any transformation on $\phi_{i'}^{ab}$ must be a unitary transformation $U^{ab, a'b'}$ inside $U(N^2)$, up to an overall factor. In particular, any *symmetry* of the system must be of this type. The symmetry of the ABJM model is $U(N) \times U(N)$, and would correspond to $U^{ab, a'b'} = f U^{aa'} V^{bb'}$.

Since the simplest type of condensed matter system is a rotationally invariant one, we should not have any angular dependence, and we should have $\phi_i^{ab} = \phi_i(\sqrt{a^2 + b^2}) = \phi_i^{ba}$. It should be then possible to diagonalize this symmetric matrix, corresponding to

considering only the constant (rotationally invariant) $m = 0$ modes for the "spherical harmonics" expansion $e^{2\pi i m \theta}$ at fixed radius $r = \sqrt{a^2 + b^2}$. In this way, only $\mathcal{O}(N)$ modes, specifically those that are spherically symmetric, out of the $\mathcal{O}(N^2)$ modes in the system are turned on. These can be thought of as the eigenvalues of ϕ_i^{ab} .

Since N is the effective maximal radius for coupling of the two fermions at different sites, it makes sense for the wavefunction in the ground state to decrease from a maximum value at $a = 1$ (neighbouring sites) to zero at $a = N$ (sites at distance N). For instance, if the wavefunction $\psi(a)$ is such that $|\psi(a)|^2 \propto N - a$, then the average distance between sites is

$$\langle a \rangle = \frac{\int |\psi(a)|^2 a (2\pi a da)}{\int |\psi(a)|^2 (2\pi a da)} = \frac{N}{2}, \quad (5.4)$$

which is consistent with having a large average distance between the electrons that couple. This form of the wavefunction, $\psi(a) \propto \sqrt{N - a}$, here just a consistent choice, is exactly what we obtain in the ABJM model. Of course, in principle, if we would be able to correctly describe the interactions between various ψ_i^{ab} , as in the ABJM model, the dynamics would select the form of $\psi(a)$. Finally, the Hubbard model field b_i must be the linear combination of the spherical modes, i.e. $b_i \sim \sum_a \psi(a) \phi_i^{aa}$.

We have already seen that to obtain the Landau-Ginzburg model from ABJM, we have only one field, χ_2 , turned on corresponding to turning on the matrix $G^2 = \sqrt{N - m} \delta_{m+1, n}$, with $(G^2 G_2^\dagger)_{mn} = (N - m) \delta_{mn}$. As in the simple model above, there are two independent rotations, in this case $U(N) \times U(N)$ rotations, acting on the indices, so the most general solution for the matrix G^2 is in fact $U(\sqrt{N - m} \delta_{mn}) V^{-1}$. We can use these to diagonalize the matrix, thus reducing the degrees of freedom turned on, from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$, as in the above condensed matter model. The ABJM field that is turned on is $\chi_2(G^2)_{mn}$, corresponding to $b_{i'} \sim \sum_a \psi_a \phi_{i'}^{aa}$.

While this field is the only one turned on in our simple toy condensed matter model, there are, in principle, many more fields. We could, for instance, have more free electrons at each site, thus having more matrix scalars, transforming in some R-symmetry group (in ABJM we have 4 complex scalars, corresponding to $\phi_1, \phi_2, \chi_1, \chi_2$, that transform under the $SU(2) \times SU(2)$ of the mass deformed ABJM). Then, we could also have matrix fermions, corresponding for instance to two electrons at site i coupling with one electron at site j , although such modes are, of course, not turned on in the Hubbard model description. To complete the field content of the ABJM model we need also the Chern-Simons gauge fields, but since those are topological and have no dynamics, we don't need to introduce any new degrees of freedom.

Chern-Simons gauge fields are, of course, no strangers to condensed matter systems, showing up, for instance, in the fractional quantum Hall effect (see for instance the

review [22]). An abelian Chern-Simons field can be obtained as follows. Consider a multi-electron wavefunction $\Psi_e(\vec{r}_1, \dots, \vec{r}_k)$ with a generic Hamiltonian

$$H_e = \sum_j \frac{|\vec{p}_j - e\vec{A}(\vec{r}_j)|^2}{2m_b} + \sum_{i < j} v(\vec{r}_i - \vec{r}_j) \quad (5.5)$$

such that $H_e \Psi_e = E \Psi_e$. We can redefine the wavefunction through the transformation

$$\Phi(\vec{r}_1, \dots, \vec{r}_k) = U \Psi_e(\vec{r}_1, \dots, \vec{r}_k) = \left[\prod_{i < j} e^{-i\tilde{\phi}\alpha(\vec{r}_i - \vec{r}_j)} \right] \Psi_e(\vec{r}_1, \dots, \vec{r}_k) \quad (5.6)$$

where $\alpha(\vec{r}_i - \vec{r}_j)$ is the angle made by $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ with a fixed axis. Since

$$U^{-1}(\vec{p}_i - e\vec{A}(\vec{r}_i))U = \vec{p}_i - e\vec{A}(\vec{r}_i) - e\vec{a}(\vec{r}), \quad (5.7)$$

where

$$e\vec{a}(\vec{r}_i) = \tilde{\phi} \sum_{j \neq i} \vec{\nabla}_i \alpha(\vec{r}_i - \vec{r}_j), \quad (5.8)$$

the new Hamiltonian reads

$$H = \sum_j \frac{|\vec{p}_j - e\vec{A}(\vec{r}_j) - e\vec{a}(\vec{r}_j)|^2}{2m_b} + \sum_{i < j} v(\vec{r}_i - \vec{r}_j) \quad (5.9)$$

so that $H\Phi = E\Phi$. Therefore after the transformation, $\vec{a}(\vec{r})$ describes a gauge field with no dynamics which, one can show is of Chern-Simons type. Such a Chern-Simons gauge field, coupled to the fermions and to the electromagnetic gauge field, plays a central role in the fractional quantum Hall effect, see e.g. [23].

A generalization of this construction to the nonabelian case is straightforward. If two fermions at sites i and i'' couple to form a boson $\phi_{i'}^{aa'}$, at site i' at the midpoint, and two other fermions at sites j and j'' couple to form a boson $\phi_{j'}^{bb'}$ at site j' at their midpoint, we can consider the field

$$e\vec{a}(\vec{r}_{i'}) = \vec{\nabla}_{i'} \sum_{j' \neq i'} \alpha(\vec{r}_i - \vec{r}_j), \quad (5.10)$$

where we have not yet specified the nonabelian indices on the gauge field. It is not hard to see that the only variable in this object is the vector $\vec{r}_{ii'} - \vec{r}_{jj'}$ (by changing the vector $\vec{r}_{ii'}$ we just produce a harmless global spatial translation in the value of the right hand side of (5.10)), as well as the discrete choice of $\vec{r}_{ii'}$ to belong to the fixed point i' or the summed point j' . Since the two vectors \vec{r} subtracted correspond to matrix indices (aa') and (bb') , we can think of this construction as giving us two nonabelian gauge fields \vec{a}^{ab} and $\hat{\vec{a}}^{a'b'}$, like the A and \hat{A} of ABJM. Moreover, the scalars $\phi_{i'}^{aa'}$ are bifundamental with respect to the two resulting gauge fields. There

remain many open problems to understand about this model, not the least of which is the symmetry group acting on the matrix Chern-Simons fields but we leave these to the interested reader. This concludes our description of the field content of ABJM and qualitative understanding of abelianization. Suffice it to say that the ABJM abelianization gives a well motivated model of AdS/CMT.

Finally, a few comments on a four dimensional picture for the Landau-Ginzburg model (5.1). The Landau-Ginzburg model makes more sense from a theoretical viewpoint as a reduction of the corresponding four dimensional theory. But here as well, the abelianization presented has in particular an ansatz with the scalar VEV b multiplying the matrix G^1 . If we had the same VEV multiplying both G^1 and G^2 , that would lead to a description of the fuzzy two-sphere, a finite N approximation of the classical two-sphere [14, 15]. As it is, we can think of the abelianization as generating a single direction, or a "fuzzy circle", therefore the resulting Landau-Ginzburg theory must also be thought of as coming from a circle reduction of a similar theory in 4 dimensions. The physical radius obtained from a fuzzy space construction was argued to be (see for instance [14])

$$R_{ph}^2 = \frac{2}{N} \text{Tr} [X^I X_I^\dagger] = \frac{2}{N} \text{Tr} [C^I C_I^\dagger] 4\pi^2 l_P^3, \quad (5.11)$$

where $l_P^3 = l_s^2 R_{11}$. Assuming this same formula holds for the less defined "fuzzy circle" case, from $\text{Tr}[G^1 G_1^\dagger] = N(N-1)/2$, we get

$$R_{ph}^2 = (N-1)|b|^2 4\pi^2 l_s^2 R_{11}. \quad (5.12)$$

If, as in the pure fuzzy sphere case, the 11th direction has radius $R_{11} = R_{ph}/k$, we obtain in the "maximally Higgs" case $|b|^2 = c^2$

$$R_{ph} = (N-1)\mu l_s^2. \quad (5.13)$$

6. Some BPS solutions with spacetime interpretation

We now return to the more general abelianization ansatz, and consider the system with $\phi_1 = \phi_2 = 0$ and gauge fields put to zero, but $\chi_1 \neq 0$ still a field (unlike in the abelian Higgs case previously described). This ansatz gives the reduced action

$$S = -\frac{N(N-1)}{2} \int d^3x \left[|\partial_\mu \chi_1|^2 + |\partial_\mu \chi_2|^2 + \frac{4\pi^2}{k^2} \left(|\chi_1|^2 (|\chi_2|^2 - c^2)^2 + |\chi_2|^2 (|\chi_1|^2 - c^2)^2 \right) \right] \quad (6.1)$$

which we now proceed to study.

6.1 Single Profile Solution

As a first pass, let's consider solutions with a single profile

$$\chi_1 = \chi_2 = f(x_1), \quad (6.2)$$

with x_1 as one of the spatial coordinates. The equation of motion for the reduced action (6.1) for this ansatz become

$$\partial_{x_1}^2 f = \frac{4\pi^2}{k^2} f \left(f^2 - \frac{\mu k}{2\pi} \right) \left(3f^2 - \frac{\mu k}{2\pi} \right), \quad (6.3)$$

from which we distill two cases:

1. Zero mass: In the massless case, $\mu = 0$, the ground state solution is simply $f(x_1) = 0$ with no other constant solutions. This is, however, not the only solution and a straightforward integration of the equation of motion yields

$$f(x_1) = \sqrt{\frac{k}{4\pi x_1}}. \quad (6.4)$$

This is a *fuzzy funnel solution* which we can check is, in fact, BPS. Indeed, the energy of solutions satisfying the above ansatz is

$$H_{\mu=0} = -N(N-1) \int dx_1 dx_2 \left[(\partial_{x_1} f)^2 + \frac{4\pi^2}{k^2} f^6 \right], \quad (6.5)$$

which, by the usual procedure of completion of squares, can be expressed as

$$H_{\mu=0} = -N(N-1) \int dx_1 dx_2 \left(\left(\partial_{x_1} f - \frac{2\pi}{k} f^3 \right)^2 + \text{surface term} \right), \quad (6.6)$$

from which we can simply read off the BPS equation

$$\partial_{x_1} f(x_1) = \frac{2\pi}{k} f(x_1)^3. \quad (6.7)$$

It is clear that this equation is solved by the fuzzy funnel solution above.

2. Nonzero mass: In this case, the constant solutions to the equations of motion are

$$f = 0, \quad f = \sqrt{\frac{\mu k}{2\pi}}, \quad f = \sqrt{\frac{\mu k}{6\pi}}. \quad (6.8)$$

Of these, only the first two are ground states. Indeed, completing squares again, we find the BPS equation

$$\partial_{x_1} f + \frac{2\pi}{k} f \left(f^2 - \frac{\mu k}{2\pi} \right) = 0, \quad (6.9)$$

from which see that indeed $f = 0$ is a trivial ground state, while the second solution ($f^2 = \mu k/2\pi$) is again the fuzzy sphere ground state. The third solution of the equations of motion ($f^2 = \mu k/6\pi$) doesn't satisfy the BPS equation, so is a non-ground state fuzzy sphere. The BPS equation has nontrivial solutions

$$f_{\mp}(x_1) = \sqrt{\frac{\mu k/2\pi}{1 \mp e^{-2\mu x_1}}} . \quad (6.10)$$

The first solution, f_- , describes a fuzzy funnel with $x_1 \in (0, +\infty)$, so f_- varies between an infinite size at $x_1 = 0$ and the fuzzy sphere ground state at $x_1 \rightarrow +\infty$,

$$f_-(0) = +\infty, \quad f_-(+\infty) = \sqrt{\frac{\mu k}{2\pi}} . \quad (6.11)$$

The second solution, f_+ , describes a fuzzy funnel with $x_1 \in (-\infty, +\infty)$, varying in size between zero at $x_1 \rightarrow -\infty$ and the fuzzy sphere at $x_1 \rightarrow +\infty$,

$$f_+(-\infty) = 0, \quad f_+(+\infty) = \sqrt{\frac{\mu k}{2\pi}} . \quad (6.12)$$

This fuzzy funnel solution will be elaborated on in the next section, where we argue that it is a generalization of the Basu-Harvey solution that describes an M2 ending on a *spherical* M5. These solutions are plotted in figure 4. below.

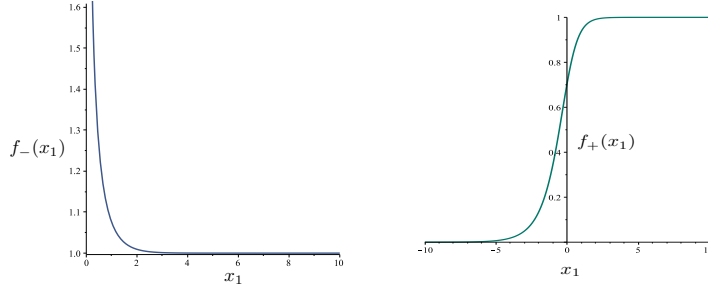


Figure 4: The normalized single-profile solutions $f_-(x_1)$ and $f_+(x_1)$

6.2 Two-Profile Solution

The above single profile solution is also fairly easily generalized to a two-profile one with

$$\chi_1 = f(x_1), \quad \chi_2 = g(x_1) . \quad (6.13)$$

With this ansatz, the equations of motion reduce to

$$\begin{aligned} \partial_{x_1}^2 f &= \frac{4\pi^2}{k^2} f \left[\left(g^2 - \frac{\mu k}{2\pi} \right)^2 + 2g^2 \left(f^2 - \frac{\mu k}{2\pi} \right) \right] , \\ \partial_{x_1}^2 g &= \frac{4\pi^2}{k^2} g \left[\left(f^2 - \frac{\mu k}{2\pi} \right)^2 + 2f^2 \left(g^2 - \frac{\mu k}{2\pi} \right) \right] . \end{aligned} \quad (6.14)$$

Again, we can complete squares in the Hamiltonian and read off the BPS equations

$$\partial_{x_1} f + \frac{2\pi}{k} f \left(g^2 - \frac{\mu k}{2\pi} \right) = 0, \quad (6.15)$$

$$\partial_{x_1} g + \frac{2\pi}{k} g \left(f^2 - \frac{\mu k}{2\pi} \right) = 0.$$

As in the single profile case above, there are again two separate cases that need to be solved separately.

1. Massless case: For $\mu = 0$, we have the solutions

$$f(x_1) = \sqrt{\frac{k}{2\pi}} \frac{C e^{-x_1}}{\sqrt{1 - C^2 e^{-2x_1}}}, \quad (6.16)$$

$$g(x_1) = \sqrt{\frac{k}{2\pi}} \frac{1}{\sqrt{1 - C^2 e^{-2x_1}}},$$

that solve both the first order BPS equations of motion as well as the general second order equations. This solution blows up at $x = \log C$, and goes to a constant in g and zero in f , corresponding to a fuzzy circle. These solutions are plotted in figure 5. below.

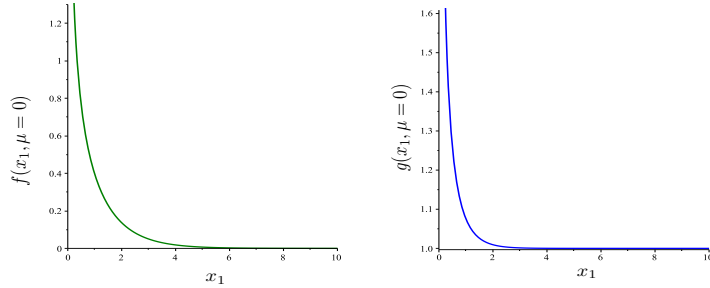


Figure 5: The normalized two-profile solutions $f(x_1)$ and $g(x_1)$

2. Nonzero mass: For $\mu \neq 0$, we have the solutions (see figure 6)

$$f(x_1) = \sqrt{\frac{\mu k}{2\pi}} \frac{C \exp \left[\mu x_1 - \frac{1}{2} e^{2\mu x_1} \right]}{\sqrt{1 - C^2 \exp(-e^{2\mu x_1})}}, \quad (6.17)$$

$$g(x_1) = \sqrt{\frac{\mu k}{2\pi}} \frac{e^{\mu x_1}}{\sqrt{1 - C^2 \exp(-e^{2\mu x_1})}}.$$

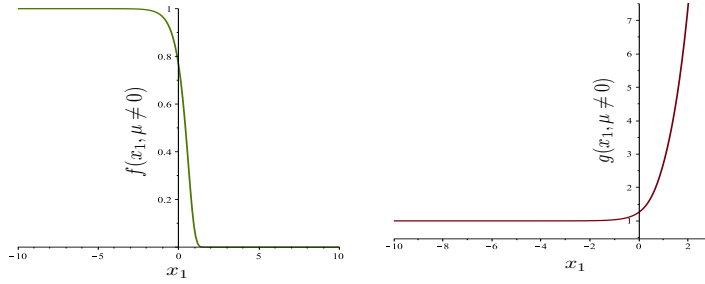


Figure 6: The normalized two-profile solutions $f(x_1)$ and $g(x_1)$ for $\mu \neq 0$.

7. Funnel solutions as M2-M5 brane systems

In this section we will try to find a spacetime interpretation for the fuzzy funnel solutions in eq. (6.12). The fuzzy funnel solution (6.4), interpolating between a sphere of infinite size and a sphere of zero size, is known to have the spacetime interpretation of a flat M2-brane ending on a flat M5-brane. From the point of view of the M2-brane theory given by the massless ABJM, the M5-brane appears as a spherical funnel solution, a M5-brane that grows from zero size at $x_1 = \infty$ to infinite size at $x_1 = 0$. We will review this case, reduced to string theory, i.e. a D2-brane ending on a D4-brane, later. Also, from the point of view of the M5-brane theory, we can write a BIon type solution, corresponding to an M2-brane growing out of the M5-brane (directions 0 and 5 are trivial, and in directions 1-4 the M2-brane appears as a BIon). From the point of view of the spacetime theory, we have an M2-M5 system preserving 1/4 supersymmetry in flat space, and we also have an M5-brane solution in the (backreacted) background of M2-branes. This picture matches nicely with the two worldvolume descriptions.

With this in mind, we expect that the fuzzy funnel solution of (6.12) should have a similar interpretation. The solution interpolating between zero and a fuzzy sphere vacuum was found in [24, 16], and we would guess that it can only match with a spacetime solution corresponding to an M2-brane ending on an M5-brane. We will see however that there is some ambiguity, related to the existence of two solutions, the one from zero to the fuzzy sphere, and one from the fuzzy sphere to infinity.

7.1 Massless case: A Fuzzy Funnel Review

The solution (6.4) corresponds in spacetime to a flat M2-brane ending on a flat M5-brane, a solution which preserves 1/4 supersymmetry as follows. In a flat background, the eleven dimensional gravitino transformation law,

$$\delta\psi_\mu = D_\mu\epsilon + \#(\Gamma^{\nu\rho\sigma\lambda}_\mu - 8\delta^\nu_\mu\Gamma^{\rho\sigma\lambda})F_{\nu\rho\sigma\lambda}, \quad (7.1)$$

must be set to zero in order to obtain a BPS solution. The M2-brane solution extended in the (0,1,2)-directions corresponds to a nonzero 3-form A_{012} , with a nonzero field strength component F_{012r} (here r is the radial part of all the coordinates transverse to the M2). The solution is given by a local supersymmetry parameter $\epsilon(r)$ which is a scalar function of r times a constant susy parameter $\epsilon_{(0)}$ satisfying

$$\Gamma^{012}\epsilon_{(0)} = \pm\epsilon_{(0)}. \quad (7.2)$$

The M5-brane solution extended in the (0,1,3,4,5,6) directions similarly gives a nonzero field strength $F_{\theta_1\dots\theta_4}$, where $\theta_1, \dots, \theta_4$ are the four angles obtained for the transverse directions (2,7,8,9,10). Again, the solution for the local supersymmetry parameter $\epsilon(r)$ is a function of r times a constant susy parameter $\epsilon_{(0)}$ satisfying

$$\Gamma^{013456}\epsilon_{(0)} = \pm\epsilon_{(0)}. \quad (7.3)$$

We can then have a solution for an M2-brane ending on an M5-brane preserving 1/4 supersymmetry by imposing both conditions (which are now compatible). We can then reduce this system to 10-dimensional string theory, thereby considering a D2-brane ending on a D4-brane.

From the point of view of the D4-brane theory in a flat background spacetime, the fuzzy funnel solution looks like a BIon-type solution. For a spacetime D2-brane in the (0,1,2)-directions, called t, x, z , and a D4-brane in the (0,1,3,4,5)-directions, with polar coordinates r, θ, ϕ for the directions (3,4,5), the worldvolume gauge field flux on the D4-brane is

$$F = (2\pi\alpha')n \sin\theta d\theta d\phi. \quad (7.4)$$

Because the solution is of the BIon type, with the D2-brane growing out of the D4-brane, we consider $z = z(r)$ on the worldvolume, leading to DBI D4-brane Lagrangian

$$\mathcal{L} = T_4 \sqrt{(1 + z'(r)^2)(r^4 + (\lambda n)^2)}, \quad (7.5)$$

where $\lambda \equiv 2\pi\alpha'$. Since \mathcal{L} is independent of z , it follows that $\partial\mathcal{L}/\partial z'$ is a constant, which we can put equal to λn , in which case we obtain

$$z' = \pm \frac{\lambda n}{r^2} \Rightarrow z = \frac{\lambda n}{r} \quad (7.6)$$

This corresponds to a funnel solution for a semi-infinite D2-brane ending on the D4-brane.

A similar story takes place for the case where there is a background created by other D2-branes (parallel with the first). It is however easier to describe what happens in the case of the type IIB solution for D3-branes ending on D5-branes (instead of D2-D4), since in that case the spacetime background is easier (there is no M theory

reduction). Consider the background generated by other D3-branes, with harmonic function $f(r)$,

$$ds^2 = f(r)^{-1/2}(-dt^2 + dx^2 + dy^2 + dz^2) + f(r)^{+1/2}(dr^2 + r^2 d\Omega_2^2 + d\bar{s}^2) \quad (7.7)$$

$$C_{(4)} = (f^{-1} - 1)dt \wedge dx \wedge dy \wedge dz$$

The DBI Lagrangean for the D5-brane reads

$$\mathcal{L} = T_5 \left[\sqrt{(1 + f'(r)z'(r)^2)(r^4 + f^{-1}(r)\lambda^2 n^2)} - \lambda n(f(r)^{-1} - 1)z'(r) \right], \quad (7.8)$$

and the same calculation leads to the same solution $z(r) = \lambda n/r$, with the function $f(r)$ dropping out completely. One can also take the near-horizon limit and consider the usual scaling $r = \alpha' U$, leading to a finite funnel solution that can be interpreted from the point of view of the D3-brane theory as

$$U(z) = \frac{n}{2\pi z}. \quad (7.9)$$

Note that if we consider a spherical D5-brane ansatz, oriented in the (t, x, y, z, Ω_2) -directions, we obtain the Lagrangian

$$\mathcal{L} = T_5 \sqrt{f^{-1}(r)(r^4 + g^{-1}(\lambda n)^2)} - \lambda n(f^{-1}(r) - 1). \quad (7.10)$$

If we take the full harmonic function $f(r) = 1 + \frac{Q}{r^4}$, there is no fixed sphere solution with $r = R = \text{constant}$, but if we drop the 1 in f , i.e. at very large r , we obtain an identity by varying with respect to r , namely $\lambda n/Q - \lambda n/Q = 0$. Therefore in flat space, we have asymptotically a solution for very large radius sphere, but at small radius we only have the funnel solution; a fixed sphere is not a solution.

7.2 Massive case: Supersymmetry and a Fluctuation Solution on the M5-brane

The mass deformation changes the 11 dimensional background spacetime from flat to [17, 19]

$$ds^2 = H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(dx_3^2 + \dots dx_{10}^2) \quad (7.11)$$

$$F_4 = 2\mu(dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 + dx^7 \wedge dx^8 \wedge dx^9 \wedge dx^{10}) + dx^0 \wedge dx^1 \wedge dx^2 \wedge dH^{-1}$$

where $H(r) = 1 - \frac{1}{4}\mu^2 r^2$. A naive guess is that the M5-brane has to live in the $(0, 1, 2, \theta, \phi, \xi)$ -directions, where θ, ϕ and ξ are the angular directions of $(3, 4, 5, 6)$, with r their radial direction, giving

$$\Gamma^{012\theta\phi\xi}\epsilon_{(0)} = \pm\epsilon_{(0)}, \quad (7.12)$$

so that the transverse M2-brane would have to be in the $(0, 1, r)$ -directions, giving

$$\Gamma^{01r}\epsilon_{(0)} = \pm\epsilon_{(0)} . \quad (7.13)$$

However we observe that the 4-form in (7.12) has nonzero F_{012r} as wanted, but since $F_{012r} = \partial_r(A_{012})$ and not $\partial_2(A_{01r})$, there must be a nontrivial Maxwell transformation that brings the gauge field A into the desired form.

How would this M2-M5-brane solution look from the point of view of the D4-brane theory (i.e., reducing to 10d string theory and focusing on the worldvolume theory)?

From the fuzzy S^2 picture in the ABJM theory an action was found for the fluctuation modes around the ground state [14]. For the scalar Φ corresponding to the fluctuations of the radius of the S^2 (transverse direction) the action reduces to just a massive mode, i.e. with potential

$$V_\Phi = \frac{1}{2} [\Phi^2 + (\nabla_{S^2}\Phi)^2] , \quad (7.14)$$

Later, in [19] the same fluctuation action was found from the DBI action of a D4-brane in the background (7.12).

For such a potential, a solution was found in [25], and was called the BIGGon (in analogy with the BIon), representing, in spacetime, an S^3 giant graviton in type IIB on the maximally supersymmetric pp wave background with F-string spikes attached at the poles. Since the pp-wave background is T-dual to (7.12) (see, for example, [17] for the explicit construction), the same solution should apply in our case. The BIGGon was found by similarly taking a single scalar field Φ on the 3-sphere (a fluctuation of the radial coordinate), with the same potential, but on S^3 instead of S^2 , i.e.

$$V_\Phi = \frac{1}{2} [\Phi^2 + (\nabla_{S^3}\Phi)^2] . \quad (7.15)$$

The solution is

$$\Phi = \frac{Q}{\sin \psi} , \quad (7.16)$$

giving the full radial coordinate (background plus fluctuation)

$$X = R \left(1 + Q \frac{g_{eff}}{\sin \psi} \right) , \quad (7.17)$$

where we have taken the S^3 parametrization to be

$$\begin{aligned} X^4 &= R \cos \psi , \\ X^3 &= R \sin \psi \cos \theta , \\ X^2 &= R \sin \psi \sin \theta \sin \phi , \\ X^1 &= R \sin \psi \sin \theta \cos \phi . \end{aligned}$$

Therefore in our case we have the BIGGon solution

$$\Phi = \frac{Q}{\sin \theta} \quad (7.18)$$

where the parametrization of the S^2 is

$$\begin{aligned} X^3 &= R \sin \theta, \\ X^4 &= R \cos \theta \sin \phi, \\ X^5 &= R \cos \theta \cos \phi. \end{aligned}$$

This solution indeed corresponds with our naive expectation of a D2-brane extending out perpendicularly from the spherical D4-brane. But the action that it extremizes corresponds to small fluctuations of the field Φ . However, in [19] it was shown how to write the full DBI action for D4-branes in the mass-deformed spacetime.

Massive case: full funnel solution

In the background (7.12) it was shown that the DBI action for the D4-brane has a fixed sphere solution, corresponding to the fuzzy sphere solution of the massive ABJM. Now we want to see if we can also find funnel solutions corresponding to the ABJM solutions (6.12) and extending the perturbative BIGGon solution above.

To this end, we consider again an M2-brane in the (0,1,2)-directions and an M5-brane in the (0,1,3,4,5,6)-directions, with (3,4,5,6) in polar coordinates r, θ, ϕ , and ξ . We reduce M-theory to type IIA on ξ and look for a D4-brane extending along 0, 1, r, θ, ϕ , with $z = z(r, \theta, \phi)$ in order to have a BIon-type solution corresponding to a perpendicular D2-brane as above.

Dimensionally reducing the background (7.12) to type IIA string theory, and writing only the terms in directions parallel to the D4-brane, we find [19]

$$\begin{aligned} C_{(5)} &= -\frac{\mu}{2kR_*} \left(\frac{H^{-1} + 1}{2} \right) dt \wedge dx \wedge z' dr \wedge r^4 d\Omega_2 + \dots \\ C_{(3)} &= (H^{-1} - 1) dt \wedge dx \wedge z' dr + \\ B &= \frac{\mu}{2kR_*} r^4 d\Omega_2 + \\ e^\phi &= \left(\frac{r}{kR_*} \right)^{3/2} H^{1/4} \end{aligned} \quad (7.19)$$

and a worldvolume flux $F = 2\lambda N d\Omega_2$, giving

$$\mathcal{F} = \lambda F - B = 2\lambda N - \frac{\mu}{2kR_*} r^4. \quad (7.20)$$

Substituting this ansatz into the action

$$S = T_4 \left[\int d^5 x e^{-\phi} \sqrt{-\det(g + \mathcal{F})} + \int (C^{(5)} + C^{(3)} \wedge \mathcal{F}) \right], \quad (7.21)$$

gives

$$S = T_4 \left\{ \int dt \wedge dx \wedge dr \wedge d\Omega_2 \sqrt{(1 + z'^2 H^{-1}) \left(1 + H^{-1} \frac{k^2 R_*^2}{r^6} \left(2\lambda N - \frac{\mu r^4}{2kR_*} \right)^2 \right)} \right. \\ \left. + \int dt \wedge dx \wedge dr \wedge d\Omega_2 \left(2\lambda N z' (H^{-1} - 1) - \frac{\mu}{2kR_*} z' \frac{H^{-1} + 1}{2} r^4 \right) \right\}. \quad (7.22)$$

Evidently, the Lagrangian \mathcal{L} is independent of z , which means that $\partial L / \partial z'$ is a constant, which we can put equal to $-T_4 2\lambda N$, in which case

$$z' = \pm \frac{\left[\frac{2\lambda N}{r^3} - \frac{\mu r}{2} + \frac{\mu^3 r^3}{16} \right] \sqrt{1 - \frac{\mu^2 r^2}{4}}}{\sqrt{1 - \frac{\mu^2 r^2}{4} - \frac{\mu^6 r^6}{4^4} + \frac{\mu^4 r^4}{4^2} - \frac{\mu^3 k R_* \lambda N}{8}}}. \quad (7.23)$$

Here we need to have $z' < 0$, since the equation is obtained by squaring an equation whose left hand side is linear in z' and has a positive coefficient, and whose right hand side is negative, after which we take the square root of z'^2 . Since $R_0^2 = 2\lambda N \mu k R_*$, after defining

$$x = \mu^2 r^2; \quad y = \mu z; \quad a = \mu R_0, \quad (7.24)$$

we obtain the equation

$$\frac{dy}{dx} = \pm \frac{\left(\frac{a^2}{x^2} - \frac{1}{2} + \frac{x}{4^2} \right) \sqrt{1 - \frac{x}{4}}}{\sqrt{1 - \frac{x}{4} - \frac{x^3}{4^4} + \frac{x^2}{4^2} - \frac{a^2}{4^2}}}. \quad (7.25)$$

However, as explained in [19], we are in the approximation $a = \mu R_0 \ll 1$, and the fixed sphere ground state solution is $r = R_0$, or $x = a^2 \ll 1$. That means that we can assume x small in the above equation, thus

$$\frac{dy}{dx} \simeq -\frac{a^2}{2x^2} + \frac{1}{4} \quad (7.26)$$

which gives finally

$$z \simeq \frac{R_0^2}{2\mu r^2} + \frac{\mu r^2}{4}. \quad (7.27)$$

This $z(r)$ has a minimum at

$$r_* = \left(\frac{4\lambda N k R_*}{\mu} \right)^{1/4} = \left(\frac{\sqrt{2} R_0}{\mu} \right)^{1/2} \gg R_0, \quad (7.28)$$

and at r_* , the minimum value of z is

$$z_{min} = \sqrt{\mu \lambda N k R_*} = \frac{R_0}{\sqrt{2}}. \quad (7.29)$$

Note that here R_0 is written in physical spacetime variables, $R_{ph}^2 = 8\pi^2 N l_P^3 f^2$, where $f = \sqrt{\mu k/(2\pi)}$ is the radius in the M2-brane worldvolume theory.

To compare with the fuzzy funnel solutions (6.12), we note that z is the equivalent of the M2-brane worldvolume direction x_1 and r is the equivalent of the transverse direction $f(x_1)$. Therefore we must consider $r(z)$ but, since it has two branches, we must choose only one. The two branches are: $r(z)$ going from 0 to r_* (for z going from ∞ to z_{min}), and $r(z)$ going from r_* to infinity. That would naively match the two solutions in (6.12), except for the fact that $r_* \gg R_0$, and $r_* \rightarrow \infty$ at $\mu \rightarrow 0$ (with N very large), whereas $R_0 = 2\mu\lambda N \rightarrow 0$ as $\mu \rightarrow 0$. In fact, for $\mu \rightarrow 0$, we obtain from (7.27)

$$r(z) = \frac{R_0}{\sqrt{2\mu z}}. \quad (7.30)$$

This can be compared with the fuzzy funnel solution (6.4), written as

$$f(x_1) = \frac{f}{\sqrt{2\mu x_1}}, \quad (7.31)$$

so the spacetime solution is a *deformation* of the $\mu = 0$ case. It also matches with the first solution in (6.12) in the $\mu \rightarrow 0$ limit. However if μ is fixed, the solution becomes (7.31) in the $x_1 \rightarrow 0$ limit, corresponding to $z \rightarrow 0$, which is not even reachable by (7.27).

It is therefore unclear to us how to relate the two branches of (7.27) to the two solutions of (6.12) precisely, other than through the general qualitative behaviour. We will leave a precise understanding of the matching to future work.

8. Conclusions

In this paper, we have studied various ansätze for abelian reductions of the ABJM model, in the general case of nonzero mass, and used them to build a better defined AdS/CMT model. We have found a general abelianization ansatz (2.22, 2.21), using the matrices G^α that describe the fuzzy funnel BPS state and fuzzy sphere ground state, and that represent a consistent truncation. A further consistent truncation led to a model with topological vortex BPS solutions, but with $|\phi| \rightarrow 0$ at both $r = 0$ and $r = \infty$ while yet another further consistent truncation led to a relativistic Landau-Ginzburg model which, depending on the parameter $c^2 = \mu k/(2\pi)$ and on the scalar vev b , extrapolates between the abelian-Higgs model, and a scalar ϕ^4 theory.

The second abelianization was used to take steps towards a better defined AdS/CMT model, since the ABJM model has a gravity dual, and the abelianization corresponds

to the collective dynamics of $\mathcal{O}(N)$ out of the $\mathcal{O}(N^2)$ fields. We also sketched a simple condensed matter model for a solid with free electrons that exhibits the same general features as the abelianization and leads to a bosonic Hubbard model, which in the continuum limit gives the relativistic Landau-Ginzburg system. It will be interesting to see if we can make the model more concrete and elaborate further on its relation to ABJM. If successful, our construction provides, in our opinion, a concrete embedding of the AdS/CMT correspondence in string theory.

In the last two sections, we studied various BPS solutions suggested by the abelianization, finding some generalizations of known solutions. We tried to find a spacetime interpretation for the BPS solutions in (6.12) as M2-M5 systems, with partial success. For small fluctuations we succeeded in matching this with the BIGGon solution for an M2 ending on a spherical M5, but for the full system we could only match only general qualitative behaviour and not the particular solution. It goes without saying that more work is needed to understand these solutions.

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References

- [1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].
- [2] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” *Prog. Theor. Phys.* **113**, 843 (2005) [hep-th/0412141].
- [3] S. S. Gubser, “Using string theory to study the quark-gluon plasma: Progress and perils,” *Nucl. Phys. A* **830** (2009) 657C [arXiv:0907.4808 [hep-th]].
- [4] D. Berenstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” *JHEP* **0204**, 013 (2002) [hep-th/0202021].
- [5] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” *JHEP* **0303**, 013 (2003) [hep-th/0212208].

- [6] R. C. Myers, S. Sachdev and A. Singh, “Holographic Quantum Critical Transport without Self-Duality,” *Phys. Rev. D* **83**, 066017 (2011) [arXiv:1010.0443 [hep-th]].
- [7] L. Huijse and S. Sachdev, “Fermi surfaces and gauge-gravity duality,” *Phys. Rev. D* **84**, 026001 (2011) [arXiv:1104.5022 [hep-th]].
- [8] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” *Class. Quant. Grav.* **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]].
- [9] C. P. Herzog, “Lectures on Holographic Superfluidity and Superconductivity,” *J. Phys. A* **42**, 343001 (2009) [arXiv:0904.1975 [hep-th]].
- [10] A. Mohammed, J. Murugan and H. Nastase, “Towards a Realization of the Condensed-Matter/Gravity Correspondence in String Theory via Consistent Abelian Truncation,” arXiv:1205.5833 [hep-th].
- [11] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” *JHEP* **0810**, 091 (2008) [arXiv:0806.1218 [hep-th]].
- [12] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk and H. Verlinde, “A Massive Study of M2-brane Proposals,” *JHEP* **0809**, 113 (2008) [arXiv:0807.1074 [hep-th]].
- [13] S. Terashima, “On M5-branes in N=6 Membrane Action,” *JHEP* **0808**, 080 (2008) [arXiv:0807.0197 [hep-th]].
- [14] H. Nastase, C. Papageorgakis and S. Ramgoolam, “The Fuzzy S^2 structure of M2-M5 systems in ABJM membrane theories,” *JHEP* **0905**, 123 (2009) [arXiv:0903.3966 [hep-th]].
- [15] H. Nastase and C. Papageorgakis, “Bifundamental fuzzy 2-sphere and fuzzy Killing spinors,” *SIGMA* **6**, 058 (2010) [arXiv:1003.5590 [math-ph]].
- [16] M. Arai, C. Montonen and S. Sasaki, “Vortices, Q-balls and Domain Walls on Dielectric M2-branes,” *JHEP* **0903**, 119 (2009) [arXiv:0812.4437 [hep-th]].
- [17] A. Mohammed, J. Murugan and H. Nastase, “Looking for a Matrix model of ABJM,” *Phys. Rev. D* **82** (2010) 086004 [arXiv:1003.2599 [hep-th]].
- [18] S. S. Gubser, “Breaking an Abelian gauge symmetry near a black hole horizon,” *Phys. Rev. D* **78**, 065034 (2008) [arXiv:0801.2977 [hep-th]].
- [19] N. Lambert, H. Nastase and C. Papageorgakis, “5D Yang-Mills instantons from ABJM Monopoles,” *Phys. Rev. D* **85** (2012) 066002 [arXiv:1111.5619 [hep-th]].
- [20] R. Auzzi and S. P. Kumar, “Non-Abelian Vortices at Weak and Strong Coupling in Mass Deformed ABJM Theory,” *JHEP* **0910**, 071 (2009) [arXiv:0906.2366 [hep-th]].

- [21] S. Sachdev, “What can gauge-gravity duality teach us about condensed matter physics?,” *Ann. Rev. Condensed Matter Phys.* **3**, 9 (2012) [arXiv:1108.1197 [cond-mat.str-el]].
- [22] S. H. Simon, ”The Chern-Simons Fermi Liquid Description of Fractional Quantum Hall States” chapter in ”Composite Fermions”, ed. O. Heinonen, World Scientific and arXiv:cond-mat/9812186
- [23] E. Witten, Lectures at the 2011 Swieca School in Campos de Jordão, Brazil
- [24] K. Hanaki and H. Lin, “M2-M5 Systems in N=6 Chern-Simons Theory,” *JHEP* **0809**, 067 (2008) [arXiv:0807.2074 [hep-th]].
- [25] D. Sadri and M. M. Sheikh-Jabbari, “Giant hedgehogs: Spikes on giant gravitons,” *Nucl. Phys. B* **687**, 161 (2004) [hep-th/0312155].